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Global Symmetry and the Branching Ratios for the K^- Capture by the Nucleon

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Branching ratios for the K^- capture by the nucleon are studied to investigate the validity of Gell-Mann's global symmetry. The following assumptions are made: 1) Gell-Mann's global symmetry is valid, 2) K-meson is pseudoscalar, 3) mass differences among baryons can be neglected and 4) processes involving K-mesons are described by treating the M. S coupling as perturbations in the lowest order that gives the effect. Under these assumptions, the matrix element for the above mentioned processes are expressed in terms of the $\pi-N$ scattering matrix elements for which experimental data can be used. The branching ratios, then, are compared with experimental results.

§ 1. Introduction

Recently Gell-Mann proposed an interesting theory concerning the connection between symmetries and strengths of the interactions.[1] The essential points of his arguments are as follows.

1. All the baryons are symmetrically coupled to the pion field with a coupling constant ~ 15—V. S coupling.

2. The K-meson couplings to the baryons are weaker and less symmetric although still charge independent—M. S coupling.

We examined how his theory predicts experimental results involving K-meson, pion and baryons under the following assumptions.

1. Gell-Mann's global symmetry is valid.

2. K-meson is pseudoscalar.

3. Mass differences between the nucleon, Λ^0 and Σ can be neglected.

4. Processes involving K-mesons are described by treating the M. S coupling as perturbations in the lowest order that gives the effect.

Under these assumptions, the matrix elements for various reactions involving K-meson, pion and baryons can be approximately expressed in terms of the $\pi-N$ scattering amplitudes. But unfortunately, besides the $\pi-N$ scattering matrix elements, there appears a matrix element for an unphysical process, which corresponds to the absorption of a fictitious pion with $I=0$. However the phase of this matrix element is expressed in terms of the $\pi-N$ scattering phase shift if we assume the invariance under the Wigner time reversal. We take the magnitude of this matrix element as an adjustable parameter.

In the present paper, we shall apply this technique to the case of the K^- capture by the nucleon. The branching ratios may be compared with experimental results by

making use of the experimental data for the $\pi-N$ scattering. In § 2 expressions are derived to be compared with experiment, in § 3 comparison is made with experiment and finally in § 4 we discuss the result thus obtained.

§ 2. Evaluation of the matrix elements for the processes $K^-+N\to Y+\pi$

It is experimentally known that there are the following modes of reactions of the K^--meson absorption by the nucleon at rest.[2]

$$\begin{aligned} K^-+p&\to \Lambda^0+\pi^0\\ &\quad\ \Sigma^0+\pi^0\\ &\quad\ \Sigma^++\pi^-\\ &\quad\ \Sigma^-+\pi^+, \end{aligned}\tag{2·1}$$

$$\begin{aligned} K^-+n&\to \Lambda^0+\pi^-\\ &\quad\ \Sigma^0+\pi^-\\ &\quad\ \Sigma^-+\pi^0. \end{aligned}\tag{2·2}$$

First we evaluate the matrix elements of these modes by treating the M. S coupling as perturbations in the lowest order that gives the effect. As an example, we take a process $K^-+p\to\Lambda^0+\pi^0$.

The K-meson interaction Hamiltonian for this case, according to the assumption (2), is given by

$$\boldsymbol{H}_K=ig_{\Lambda K}\bar{\Lambda}^0\gamma_5 pK^-+ig_{\Sigma k}\bar{\Sigma}^0\gamma_5 pK^-+ig_{\Sigma K}\sqrt{2}\,\bar{\Sigma}^-\gamma_5 nK^-\tag{2·3}$$

which may be rewritten as follows.

$$\boldsymbol{H}_K=i(g_{\Lambda K}+g_{\Sigma K})/\sqrt{2}\,\bar{Z}^0\gamma_5 pK^-+i(g_{\Lambda K}-g_{\Sigma K})/\sqrt{2}\,\bar{Y}^0\gamma_5 pK^-+ig_{\Sigma K}\sqrt{2}\,\bar{\Sigma}^-\gamma_5 nK^-\tag{2·4}$$

where

$$\Lambda^0=(Z^0+Y^0)/\sqrt{2}\quad\text{and}\quad\Sigma^0=(Z^0-Y^0)/\sqrt{2}\,.$$

Notations used here are the same as those in Gell-Mann's paper.[1]

The above mentioned process is given by the matrix element

$$\begin{aligned}\langle\Lambda^0\pi^0|S|K^-p\rangle&=\left\langle\Lambda^0\pi^0\left|\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int\cdots\int dt_1\cdots dt_n P[H(t_1),\ \cdots,\ H(t_n)]\right|K^-p\right\rangle\\ &=(-i)\left\langle\Lambda^0\pi^0\left|\int\frac{d^4x}{\sqrt{2\omega_k}}e^{ikx}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int\cdots\int dt_1\cdots dt_n\right.\right.\\ &\quad\left.\left.\times P[H(t_1),\ \cdots H(t_n)O(x)]\right|p\right\rangle\end{aligned}\tag{2·5}$$

where

$$H(t)=\int[\boldsymbol{H}_K(x)+\boldsymbol{H}_\pi(x)]d^3x$$

and

$$O=i(g_{\Lambda K}+g_{\Sigma K})/\sqrt{2}\,\bar{\Xi}^0\gamma_5 p+i(g_{\Lambda K}-g_{\Sigma K})/\sqrt{2}\,\bar{Y}^0\gamma_5 p+i\sqrt{2}\,g_{\Sigma K}\bar{\Sigma}^-\gamma_5 n.$$

This expression can be transformed to the Heisenberg representation as[3]

$$\langle\Lambda^0\pi^0|S|\,K^-p\rangle=(-i)\left\langle\Lambda^0\pi^0\left|\int\frac{d^4x}{\sqrt{2\omega_k}}e^{ikx}\boldsymbol{O}\right|\boldsymbol{p}\right\rangle \tag{2·6}$$

where $|\Lambda^0\rangle$ or $|\boldsymbol{p}\rangle$ represents the physical particle with complete K-meson and pion clouds around it, $\boldsymbol{O}(x)$ is the operator in the Heisenberg representation and k_μ is the momentum of the incident K^--meson. The above expression is exact. We next rewrite (2·6) as follows, using the assumptions (1), (3) and (4),

$$\begin{aligned}\langle\Lambda^0\pi^0|S|\,K^-p\rangle&=-i/\sqrt{2}\int\frac{d^4x}{\sqrt{2\omega_k}}e^{ikx}[\langle\boldsymbol{\Xi}^0\pi^0|\boldsymbol{O}(x)\,|\,\boldsymbol{p}\rangle+\langle\boldsymbol{Y}^0\pi^0|\boldsymbol{O}(x)\,|\,\boldsymbol{p}\rangle]\\&=-i/\sqrt{2}\int\frac{d^4x}{\sqrt{2\omega_k}}e^{ikx}[\langle\boldsymbol{p}\pi^0|\boldsymbol{O}'(x)\,|\,\boldsymbol{p}\rangle+\langle\boldsymbol{n}\pi^0|\boldsymbol{O}'(x)\,|\,\boldsymbol{p}\rangle]\end{aligned} \tag{2·7}$$

where $\boldsymbol{O}'(x)$ corresponds to the operator $\boldsymbol{O}(x)$ in which Ξ^0, Y^0 and Σ^- are replaced by p, n and n respectively. Or, explicitly we may write

$$\begin{aligned}\boldsymbol{O}'(x)=&\,i(g_{\Lambda K}+g_{\Sigma K})/\sqrt{2}\,\bar{\boldsymbol{N}}\gamma_5(1+\tau_3)/2\boldsymbol{N}+i(g_{\Lambda K}-g_{\Sigma K})/\sqrt{2}\,\bar{\boldsymbol{N}}\gamma_5\tau^-\boldsymbol{N}\\&+i\sqrt{2}\,g_{\Sigma K}\bar{\boldsymbol{N}}\gamma_5(1-\tau_3)/2\boldsymbol{N}\end{aligned} \tag{2·8}$$

where the operator $\boldsymbol{N}$ represents the nucleon wave operator in the Heisenberg representation.

Now, let us examine one of these terms in detail, say $(-i)\langle\boldsymbol{p}\pi^0|\int d^4x/\sqrt{2\omega_K}\,e^{ikx}\bar{\boldsymbol{N}}\gamma_5\tau_3\boldsymbol{N}|\,\boldsymbol{p}\rangle$. As was shown by Low,[3] this quantity, on the energy shell, is essentially the $\pi-N$ scattering matrix element $\langle\pi^0p\,|S|\,\pi^0p\rangle$, that is to say,

$$(-i)\left\langle\boldsymbol{p}\pi^0\left|\int\frac{d^4x}{\sqrt{2\omega_k}}e^{ikx}\bar{\boldsymbol{N}}\gamma_5\tau_3\boldsymbol{N}\right|\boldsymbol{p}\right\rangle=\frac{1}{g_\pi}\langle\pi^0p\,|S|\,\pi^0p\rangle \tag{2·9}$$

where $k_\mu{}^2+\mu^2=0$, μ: pion mass and g_π is the coupling constant of the $\pi-N$ interaction. In our case, however, k_μ is the momentum of the incident K^--meson, so the matrix element (2·9) describes the off-the-energy-shell process. Nevertheless we can make the connection between this and the actual process, since baryons are heavy compared with mesons and we may neglect the spatial momentum dependence as far as the s-wave is concerned. Then eq. (2·9) is valid approximately even for this case. As for the treatment of the p-wave, we shall state briefly in § 3.

Another term concerned here is

$$(-i)\left\langle\boldsymbol{p}\pi^0\left|\int\frac{d^4x}{\sqrt{2\omega_k}}e^{ikx}\bar{\boldsymbol{N}}\gamma_5\boldsymbol{N}\right|\boldsymbol{p}\right\rangle.$$

This corresponds to the absorption of a fictitious pion with $I=0$. (We shall call this fictitious particle as σ-particle hereafter.) So we may write this as

$$(-i)\left\langle p\pi^0\right|\int\frac{d^4x}{\sqrt{2\omega_k}}e^{ikx}\bar{N}\gamma_5 N\left|p\right\rangle=\frac{1}{g_\sigma}\langle p\pi^0|S|p\sigma\rangle \tag{2·10}$$

where g_σ is the σ-particle coupling constant, which we assume to be so small that g_σ^2 can be neglected. As this matrix element does not correspond to the experimentally measurable quantity, we take this term as an adjustable parameter. In general this term will be complex, so there are two independent parameters. However, as we shall see below, the phase of this matrix element is expressed in terms of the $\pi-N$ scattering phase shift if we assume the invariance under the Wigner time reversal. Thus, two independent parameters could be reduced to one. This follows from the fact that the reaction matrix K is real and symmetric if K matrix is invariant under the Wigner time reversal.[5]

For the s-wave, K matrix has two channels for $I=1/2$ submatrix and is expressed as

$$\mathrm{K}=\begin{pmatrix}0 & \alpha\\ \alpha & \tan\delta_1\end{pmatrix}\begin{matrix}\cdots\sigma N\\ \cdots\pi N\end{matrix}$$

where α is real. Then the T-matrix is given by

$$\mathrm{T}=\begin{pmatrix}0 & e^{i\delta_1}\cos\delta_1\cdot\alpha\\ e^{i\delta_1}\cos\delta_1\cdot\alpha & e^{i\delta_1}\sin\delta_1\end{pmatrix}. \tag{2·11}$$

Thus, we see the phase of the scattering matrix for the processes $\sigma+p\to p+\pi^0$ is given by δ_1.

Finally, using eqs. (2·9) and (2·10), eq. (2·6) may be expressed as

$$\langle\Lambda^0\pi^0|S|K^-p\rangle=G_1/4\{\langle p\pi^0|S|p\pi^0\rangle+\sqrt{2}\langle n\pi^0|S|p\pi^-\rangle\}+G_2/4\langle p\pi^0|S|p\sigma\rangle \tag{2·12}$$

where

$$G_1=(g_{\Lambda K}-g_{\Sigma K})/g_\pi$$
$$G_2=(g_{\Lambda K}+3g_{\Sigma K})/g_\pi.$$

Similarly, the matrix elements for other processes are expressed in terms of the $\pi-N$ scattering amplitudes as follows:

$$\begin{aligned}
&\langle\Sigma^0\pi^0|S|K^-p\rangle=G_1/4\{\langle p\pi^0|S|p\pi^0\rangle-\sqrt{2}\langle n\pi^0|S|p\pi^-\rangle\}+G_2/4\langle p\pi^0|S|p\sigma\rangle,\\
&\langle\Sigma^+\pi^-|S|K^-p\rangle=G_1/2\langle p\pi^-|S|p\pi^-\rangle,\\
&\langle\Sigma^-\pi^+|S|K^-p\rangle=G_1/2\sqrt{2}\langle n\pi^+|S|p\pi^0\rangle+G_2/2\sqrt{2}\langle n\pi^+|S|p\sigma\rangle,
\end{aligned} \tag{2·13}$$

$$\begin{aligned}
&\langle\Lambda^0 n^-|S|K^-n\rangle=G_1/4\{\langle p\pi^-|S|p\pi^-\rangle+\sqrt{2}\langle n\pi^-|S|n\pi^-\rangle\}+G_2/4\langle p\pi^-|S|n\sigma\rangle,\\
&\langle\Sigma^0\pi^-|S|K^-n\rangle=G_1/4\{\langle p\pi^-|S|n\pi^0\rangle-\sqrt{2}\langle n\pi^-|S|n\pi^-\rangle\}+G_2/4\langle p\pi^-|S|n\sigma\rangle,
\end{aligned} \tag{2·14}$$

$$\langle \Sigma^- \pi^0 | S | K^- n \rangle = G_1/2\sqrt{2}\,\langle n\pi^0 | S | n\pi^0 \rangle + G_2/2\sqrt{2}\,\langle p\pi^- | S | n\sigma \rangle .$$

Introducing the eigenamplitudes of the isospin for the $\pi-N$ scattering, we may write these as

$$\begin{aligned}
&\langle \Lambda^0 \pi^0 | S | K^- p \rangle = G_1/12\,(4M_3 - M_1) + \sqrt{3}\,G_2/12\,M_\sigma ,\\
&\langle \Sigma^0 \pi^0 | S | K^- p \rangle = G_1/4\,M_1 + \sqrt{3}\,G_2/12\,M_\sigma ,\\
&\langle \Sigma^+ \pi^- | S | K^- p \rangle = G_1/6\,(2M_1 + M_3), \qquad (2\cdot 15)\\
&\langle \Sigma^- \pi^+ | S | K^- p \rangle = G_1/6\,(M_1 - M_3) + \sqrt{3}\,G_2/6\,M_\sigma ,\\
&\langle \Lambda^0 \pi^- | S | K^- n \rangle = G_1/6\sqrt{2}\,(4M_3 - M_1) + G_2/2\sqrt{6}\,M_\sigma ,\\
&\langle \Sigma^0 \pi^- | S | K^- n \rangle = -G_1/6\sqrt{2}\,(M_1 + 2M_3) - G_2/2\sqrt{6}\,M_\sigma , \qquad (2\cdot 16)\\
&\langle \Sigma^- \pi^0 | S | K^- n \rangle = G_1/6\sqrt{2}\,(M_1 + 2M_3) + G_2/2\sqrt{6}\,M_\sigma
\end{aligned}$$

where

$$\begin{aligned}
&M_1 = \langle I = \tfrac{1}{2} | S | I = \tfrac{1}{2} \rangle\\
&M_3 = \langle I = \tfrac{3}{2} | S | I = \tfrac{3}{2} \rangle\\
&M_\sigma = \langle I = \tfrac{1}{2}(N, \pi) | S | I = \tfrac{1}{2}(N, \sigma) \rangle .
\end{aligned}$$

Note that between these matrix elements there are the following relations owing to the charge independence.

$$\begin{aligned}
&\langle \Sigma^- \pi^+ | S | K^- p \rangle + \langle \Sigma^+ \pi^- | S | K^- p \rangle = 2\,\langle \Sigma^0 \pi^0 | S | K^- p \rangle ,\\
&\langle \Sigma^0 \pi^- | S | K^- n \rangle + \langle \Sigma^- \pi^0 | S | K^- n \rangle = 0, \qquad (2\cdot 17)\\
&\sqrt{2}\,\langle \Lambda^0 \pi^0 | S | K^- p \rangle = \langle \Lambda^0 \pi^- | S | K^- n \rangle .
\end{aligned}$$

§ 3. Comparison with experiment

Experiments on the processes (2·1) and (2·2) have been done by means of the hydrogen bubble chamber and the emulsion, respectively.[2)]

As the processes (2·2) are studied using bound neutrons, their measurement is more difficult and uncertain. So, we first examine the branching ratios for the processes (2·1) : i.e., $K^- + p \to Y + \pi$. In these processes, K^--mesons are absorbed at rest, and we may expect that the K^--mesons are absorbed from the s-state.

In this case, we may write eq. (2·15) as

$$M_1 = \frac{1}{k} e^{i\delta_1} \sin\delta_1 ; \quad M_3 = \frac{1}{k} e^{i\delta_3} \sin\delta_3 ;$$

$$M_\sigma = \frac{1}{k} e^{i\delta_1} \cos\delta_1 \cdot \alpha \qquad (3\cdot 1)$$

where δ_1 and δ_3 are the s-wave $\pi-N$ scattering phase shifts in the states of isospin $\frac{1}{2}$ and $\frac{3}{2}$, respectively.

As we have neglected the mass difference between Λ^0 and Σ, it is not certain which energy of the pion we should take to evaluate the value of δ_1 and δ_3. For example, the outgoing pion energy is about 80 Mev for the processes $K^-+N\to\Sigma+\pi$ and is about 150 Mev for the processes $K^-+N\to\Lambda^0+\pi$. If the incident K^--meson energy is transferred to the corresponding pion energy, the incident pion kinetic energy would be about 350 Mev. So, we varied the energy from 100 Mev to 400 Mev, and at each energy examined the branching ratios. To evaluate the value of δ_1 and δ_3, we used Orear's experimental formula[6]

$$\begin{aligned}\delta_1&=(0.167\pm0.012)\eta,\\ \delta_3&=(-0.105\pm0.010)\eta\end{aligned}\tag{3·2}$$

which seem to hold up to fairly high energy.*

Then, from eq. (2·15), we have

$$\begin{aligned}|\langle\Lambda^0\pi^0|S|K^-p\rangle|^2&=C[I_\Lambda+II_\Lambda\zeta+III_\Lambda\zeta^2],\\ |\langle\Sigma^0\pi^0|S|K^-p\rangle|^2&=C[I_{\Sigma^0}+II_{\Sigma^0}\zeta+III_{\Sigma^0}\zeta^2],\\ |\langle\Sigma^+\pi^-|S|K^-p\rangle|^2&=C[I_{\Sigma^+}+II_{\Sigma^+}\zeta+III_{\Sigma^+}\zeta^2],\\ |\langle\Sigma^-\pi^+|S|K^-p\rangle|^2&=C[I_{\Sigma^-}+II_{\Sigma^-}\zeta+III_{\Sigma^-}\zeta^2]\end{aligned}\tag{3·3}$$

where

$$C=\left(\frac{G_1}{12}\right)^2,\quad \zeta=\sqrt{3}\cdot\frac{G_2}{G_1}\cdot\alpha\quad(G_1\neq0).$$

If $g_{\Lambda K}=g_{\Sigma K}$, as is easily seen from eq. (2·15), the process $K^-+p\to\Sigma^++\pi^-$ is forbidden. It will be noticed that this is independent of the assumption (4). As we shall see later, Tiomno's theory of Hyperons and K-mesons[7] is not satisfactory, because the experimentally established reaction $K^-+p\to\Sigma^++\pi^-$ is also forbidden for the same reason mentioned above.

Now, we see from eq. (3·3) that there remains essentially one parameter ζ, as far as we examine the branching ratios. This seems satisfactory for testing the global symmetry. We first adjust the parameter ζ to minimize the process $K^-+p\to\Lambda^0+\pi^0$, which is experimentally known to be very rare. Results are summarized in Table I.

It is known from experiment that the branching ratios are about $2:1:1:0.25$.[8] From the Table I, we see that apart from the qualitative feature that the process $K^-+p\to\Sigma^-+\pi^+$ is dominant, the results are not very good. Next, we try to adjust ζ so as to give an overall fit for the whole processes. Main features are reproduced in Table II, including the processes (2·2).

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* E. g. Proceedings of the 7th Rochester Conference. Introductory talk by Cassels. An experimental analysis by Russian work shows that at least δ_3 seems linear up to 300~400 Mev. This is reasonable since the virtual states involved have nucleon pairs with high virtual energies and the range of interaction seems extremely short. For the same reason, we assume the linear character for the δ_1 phase shift.

Table I

	δ_1	δ_3	I_Λ	II_Λ	III_Λ	I_{Σ^0}	II_{Σ^0}	III_{Σ^0}
$E=100$ (Mev)	0.119	−0.075	0.378	1.85	2.24	0.243	−1.61	2.24
150	0.177	−0.114	0.567	1.76	1.40	0.402	−1.58	1.40
200	0.237	−0.149	0.693	1.58	0.993	0.490	−1.43	0.993
300	0.356	−0.222	0.897	1.30	0.562	0.702	−1.21	0.562
400	0.474	−0.298	1.03	1.01	0.352	0.825	−1.18	0.352

I_{Σ^+}	II_{Σ^+}	III_{Σ^+}	I_{Σ^-}	II_{Σ^-}	III_{Σ^-}	ζ	$\Sigma^- : \Sigma^+ : \Sigma^0 : \Lambda^0$
0.286	0	0	0.338	−3.59	8.95	0.441	14 : 1 : 5 : 0.05
0.355	0	0	0.476	−3.25	5.62	0.610	14 : 1 : 6 : 0.05
0.437	0	0	0.577	−2.98	3.97	0.792	12 : 1 : 5 : 0.15
0.955	0	0	0.716	−2.58	2.24	1.16	9 : 1 : 3 : 0.15
0.923	0	0	0.807	−2.08	1.41	1.44	8 : 1 : 3 : 0.3

Note: Normalization is made such that the mode $K^-+p\to\Sigma^++\pi^-$ is unity.

Table II

E (Mev)	ζ	Σ^- :	Σ^+ :	Σ^0 :	Λ^0 :	Σ^- :	Σ^0 :	Λ^0
100	0.15	4 :	1 :	2.0 :	0.6 :	0.85 :	0.85 :	1.2
150	0.20	4 :	1 :	2.2 :	0.8 :	1 :	1 :	1.6
200	0.30	4.2 :	1 :	2.8 :	0.7 :	0.57 :	0.57 :	1.4
300	0.40	2.2 :	1 :	1.4 :	0.5 :	0.25 :	0.25 :	0.96
400	0.40	1.9 :	1 :	1.5 :	0.8 :	0.26 :	0.26 :	1.6
experiment[9]		2 :	1 :	$1\frac{1}{4}$:	0 :	0.5 :	0.5 :	0

From the Table II, it is seen that the branching ratios for the process $K^-+N\to\Sigma+\pi$ are not inconsistent with the experimental results. However, the processes $K^-+N\to\Lambda^0+\pi$ is more than several times larger than the experimental results.

So far, we have confined ourselves to the s-wave, but for the pion energy $E=100$ Mev$\sim$200 Mev, the p-wave will contribute considerably owing to the $(3\cdot3)$-state resonance. So we must estimate the contribution from the p-wave part. In this case, eq. $(3\cdot1)$ has an additional contribution

$$M_{1p}=0$$

$$M_{3p}=\frac{p}{k^2}\{2e^{i\delta_{33}}\sin\delta_{33}\cos\theta-e^{i\delta_{33}}\sin\delta_{33}\sin\theta\, e^{i\varphi}\} \qquad (3\cdot4)$$

where $k_\mu{}^2+\mu^2=0$ and $p_\mu{}^2+m^2=0$; m: K-meson mass. In transferring from the π-scattering to the K^--absorption, we have assumed a linear dependence on the spatial momentum since p-wave is involved. Using eq. $(3\cdot4)$, we have examined the contribution from the p-wave and it turned out that the result became still worse than that with

the s-wave only, ratios being $1:1:0:4$. From this fact, we may conclude that the K^--meson is absorbed mainly from the s-state.

§ 4. Concluding Remarks

In § 2 and § 3, we have examined the branching ratios for the processes (2·1) and (2·2), considering that Gell-Mann's global symmetry may predicts at least qualitative features of the experiment. The result shows that the branching ratios for the processes $K^-+N\to\Sigma+\pi$ are fairly well fitted to the experiment for rather high pion energies 300 Mev$\sim$400 Mev. It is known from the phenomenological analysis that the phase difference between matrix elements of the $I=0$ and $I=1$ must be large, i. e. 70°, to be fitted to the branching ratios for the processes $K^-+N\to\Sigma+\pi$,[10] which corresponds to the fact that for rather high pion energies we could fit them well. The situation is not so well for the processes $K^-+N\to\Lambda^0+\pi$ as the processes $K^-+N\to\Sigma+\pi$. In general it seems difficult to forbid the process $K^-+p\to\Lambda^0+\pi^0$, allowing the process $K^-+p\to\Sigma^0+\pi^0$.

In this respect, Tiomno's theory of Hyperons and K-mesons is also unsatisfactory, as already mentioned. In his theory, both J_3 and J_3' (see his paper[7]) are conserved in the processes (2·1) and (2·2). But if this is the case, the process $K^-+p\to\Sigma^++\pi^-$ is forbidden, because J_3' is not conserved. And his theory predicts the branching ratios $\Sigma^-:\Sigma^+:\Sigma^0:\Lambda^0$ as $2:0:1:1$, which are not consistent with experimental results.

As a final remark, let us estimate the magnitude of the K-meson coupling constant. For this purpose, we must know the absolute value of the cross section for each process. Experiments show that for the processes $K^-+p\to\Sigma^\pm+\pi^\mp$ the energy dependence of the cross section obeys the $1/v$ law.[11] At $E=40$ Mev, σ^{tot} is about 10 mb. From this and Table II, taking the value of the outgoing pion energy E as 100 Mev, we obtain the result

$$(g_{\Lambda K}-g_{\Sigma K})^2/4\pi\sim 4.$$

It will be noticed that the magnitudes of the K-couplings are not separately obtained from these treatments. If $g_{\Lambda K}^2/4\pi$ is of the order of about unity, as Gell-Mann pointed out, this result shows that

$$1\lesssim g_{\Sigma K}^2/4\pi\lesssim 10.$$

We could not, of course, decide only from these results whether Gell-Mann's global symmetry is valid or not. Many attempts will further be needed to decide the validity of the idea of global symmetry from various points of view.

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Phenomenological Model on the Interactions of Elementary Particles, II* **

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Many authors have shown that the universal V-A coupling of the Fermi interaction is not inconsistent with various experimental results. Their assumptions are more or less ad hoc. In this paper, it will be shown phenomenologically that the universal V-A coupling is a natural and almost unique consequence of our previously mentioned standpoint; especially, a possibility of S-T-P coupling of β-decay interaction is definitely excluded. To lead to this end, we use one additional assumption setting the relation between a weak boson-fermion interaction and our fundamental interaction.

§ 1. Introduction

Our basic standpoint in I[1] was to admit the common character of the interactions belonging to each stratum of interaction strength. This standpoint was expressed by four assumptions:

I) Species ν, e, μ, n, p and Λ are assumed to be elementary or different particles and other stronglies are composed of n, p, Λ and their antiparticles.

II) The original interactions are of the Fermi type without derivative.

III) All interactions are invariant under the individual Γ_5 transformation.***

IV) The interaction $g(\bar{\psi}_a\, o\, \psi_b)(\bar{\psi}_c\, o'\, \psi_d)$ is strong if $a=b$ and $c=d$, and weak if otherwise. Such combination as $a=b$ and $c=d$, if they contain leptons, are missing or much weaker than the weak interactions for some unknown reasons.

The most noticeable implication of above four assumptions is the parity nonconservation in all weak interactions. As is well known, it was established that the Λ-decay displays the parity nonconservation.[2] Because this process contains no neutrino, the parity nonconservation is not the characteristic property of the process with neutrino but

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* The preceding article, published in Prog. Theor. Phys. **19** (1958), 475, is quoted as I.

** The preliminary account of this paper was published in Prog. Theor. Phys. **19** (1958), 589 (L).

*** The individual Γ_5 transformation of "a"-field is defined as

$$\psi_a \to \eta_a{}^{\alpha}\, \gamma_5\, \psi_a,\ \psi_b \to \eta_b{}^{\alpha}\, \psi_b,\quad \psi_c \to \eta_c{}^{\alpha}\, \psi_c, \cdots$$

$$m_a \to -m_a,\qquad m_b \to m_b,\qquad m_c \to m_c, \cdots$$

where η's are phase factors: $|\eta|^2=1$.

is common in all weak processes. The result of this experiment ensures the validity of our standpoint.

Recent experiments give some information on the asymmetry and polarization of particles emitted in decay processes. Many authors obtained the consistency of the assumption of universal V–A coupling of the Fermi interaction with these experimental results.[3)4)5)] But their assumption is ad hoc. In this note, we show phenomenologically that the universal V–A coupling* of the Fermi interaction containing leptons, assumed by many authors, is necessary from our standpoint.

From our point of view, the Fermi interaction is the fundamental one and others should be understood as an apparent form. Since we have no knowledge about the structure of a boson as a composite state of fermions, we must assume some relation between the fundamental interaction and others, as will be shown in § 2.

In § 3, it will be shown that our assumption III) gives mutual relation of phases, η's, and only those cases in which β-decay interaction is of V-A type are allowed but all other cases in which β-decay interaction is of S–T–P type are inconsistent with the experimental fact: the preferential backward emission of e^+ from μ^+ meson.

§ 2. The boson-fermion interaction as an effective form of the Fermi interaction

According to our assumption II) mentioned in § 1, weak boson-fermion interactions should be considered to be an effective one induced by fundamental Fermi interactions. So, we treat the decay processes containing a boson as follows. The effective transition matrix element of decay such as $c \to a + \bar{b}$ induced by the V–A type Fermiinteraction, which is restricted by the assumption III), is assumed to be written as

$$M = (2\pi)^4 \delta^4(k-p-q)\,(\bar{\psi}_a(p)\,(1 \pm f\gamma_5)\gamma_\rho \psi_b(q))\cdot \varphi_c(k)\cdot A\cdot k_\rho, \qquad (2\cdot 1)$$

where $f \simeq 1$.[6)] This assumption is tentatively applied to processes without leptons, i.e., Λ-decay process. This form of M is formally equivalent to that of gradient weak boson-fermion interaction. The factor A, an effective coupling parameter, may be a function of masses of participating particles and has the dimention of length in natural unit.

§ 3. On the uniqueness of the universal V-A coupling of the Fermi interaction containing leptons

We have acquired some experimental facts concerning the asymmetry and polarization of particles emitted in decay processes.

---

* It is to be noticed to which ordering of four fermion fields we refer as the V–A type Fermi interaction. From the assumption III), it is possible to forbid the mixing V–A and S–T–P among the weak interactions under consideration. The order of four fermion fields is taken so as to allow such a separation.

Especially important facts are the following:

a) The e^- (e^+) emitted from a polarized nucleus is almost completely antipolarized (polarized) along the direction of its motion.[7]

b) The e^+ emitted in $\pi^+-\mu^+-e^+$ chain predominates backwards relative to the direction of motion of μ^+ meson.[8]

b_1) The e^+ emitted in this chain is almost completely polarized along the direction of its motion, according to the experimental result obtained by Liverpool group.[8]

b_2) While, according to the experimental result obtained by Columbia group, the direction of polarization of e^+ is contrary to that obtained by Liverpool group.[9]

c) The polarization of μ^+ emitted from π^+ is not yet known.

d) The similar situation as $\pi^+-\mu^+-e^+$ chain is also realized in $K^+-\mu^+-e^+$ chain.[10]

e) Most π mesons are emitted preferentially in the direction of $(\boldsymbol{p}_{\pi^-}\times\boldsymbol{p}_\Lambda)$ in $\Lambda\to p+\pi^-$ decay, and the asymmetry coefficient $\alpha_{\text{observed}}\simeq(0.44\sim0.52)$.[2]

Using our assumptions, we will investigate which combination of weak interactions is consistent with the experiments mentioned above. It will be seen that the interaction of β-decay, μ-capture and $\Lambda\to p+\mu^-(\text{or } e^-)+\nu+p$ can be simultaneously of either V–A or S–T–P type. If we assume that these interactions are of V–A type and admit the experimental facts a), the possible combinations of weak interactions are limited to the following cases. (See Appendix 1.)

$$
\begin{aligned}
&\text{i)} && f(\bar\psi_p(1\pm\gamma_5)\gamma_\rho\,\psi_n)\,(\bar\psi_e(1-\gamma_5)\gamma_\rho\,\psi_\nu'),\\
&\text{ii)} && f(\bar\psi_e(1-\gamma_5)\gamma_\rho\,\psi_\mu)\,(\bar\psi_\nu\,\gamma_\rho'\,\psi_\nu'),\\
&\text{iii)} && f(\bar\psi_p(1\pm\gamma_5)\gamma_\rho\,\psi_n)\,(\bar\psi_\mu(1-\gamma_5)\gamma_\rho\,\psi_\nu),\\
&\text{iv)} && f(\bar\psi_\Lambda(1\pm\gamma_5)\gamma_\rho\,\psi_n)\,(\bar\psi_p\,\gamma_\rho'\,\psi_p),\\
&\text{v)} && f(\bar\psi_p(1\pm\gamma_5)\gamma_\rho\,\psi_\Lambda)\,(\bar\psi_e(1-\gamma_5)\gamma_\rho\,\psi_\nu'),\\
&\text{vi)} && f(\bar\psi_p(1\pm\gamma_5)\gamma_\rho\,\psi_\Lambda)\,(\bar\psi_\mu(1-\gamma_5)\gamma_\rho\,\psi_\nu),
\end{aligned}
\qquad (3\text{–}1)
$$

i), ii) and iii) are as same as those in (3-1), and

$$
\begin{aligned}
&\text{iv)} && \sum_{S,T,P} f(\bar\psi_\Lambda(1\pm\gamma_5)\,o\,\psi_n)\,(\bar\psi_p\,o\,\psi_p),\\
&\text{v)} && f(\bar\psi_p(1\mp\gamma_5)\gamma_\rho\,\psi_\mu)\,(\bar\psi_e(1-\gamma_5)\gamma_\rho\,\psi_\nu'),\\
&\text{vi)} && f(\bar\psi_p(1\mp\gamma_5)\gamma_\rho\,\psi_\Lambda)\,(\bar\psi_\mu(1-\gamma_5)\gamma_\rho\,\psi_\nu),
\end{aligned}
\qquad (3\text{–}2)
$$

i), iv) and v) are as same as those in (3–1), and

$$
\begin{aligned}
&\text{ii)} && \sum_{S,T,P} f(\bar\psi_e(1-\gamma_5)\;0\;\psi_\mu)\,(\bar\psi_\nu\;0\;\psi_\nu'),\\
&\text{iii)} && f(\bar\psi_p(1\pm\gamma_5)\gamma_\rho\,\psi_n)\,(\bar\psi_\mu(1+\gamma_5)\gamma_\rho\,\psi_\nu),\\
&\text{vi)} && f(\bar\psi_p(1\pm\gamma_5)\gamma_\rho\,\psi_\Lambda)\,(\bar\psi_\mu(1+\gamma_5)\gamma_\rho\,\psi_\nu),
\end{aligned}
\qquad (3\text{–}3)
$$

and

i), ii) and iii) are as same as those in (3–3),

v) is as same as that in (3–2), and

iv) $\sum_{S,T,P} f(\bar{\psi}_\Lambda(1\mp\gamma_5)o\ \psi_n)\ (\bar{\psi}_p\ o\ \psi_p)$,

vi) $f(\bar{\psi}_p(1\mp\gamma_5)\gamma_\rho\ \psi_\Lambda)\ (\bar{\psi}_\mu(1+\gamma_5)\gamma_\rho\ \psi_\nu)$, (3–4)

where the double signs are taken with the same order, $\psi_\nu'=\psi_\nu$ or $\psi_\nu^c(=C^{-1}\bar{\psi}_\nu)$, $(C\gamma_\mu C^{-1}=-\gamma_\mu^T,\ C^+=C^{-1}=-C^*)$, $\gamma_\rho'=\gamma_\rho$ or $\gamma_5\gamma_\rho$ and ψ_μ is the annihilation operator of μ^-.

Here we assume that the lepton number conservation law is valid. Then, if $\psi_{\prime\nu}=\psi_\nu$, we should assign the same lepton number to μ^- and e^-, say, $+1$, and if $\psi_\nu'=\psi_\nu^c$, we should assign the same lepton number to μ^- and e^+ say, $+1$. The expressions (3–1) to (3–4) are irrespective of the lepton number assignment. (See Appendix 1.)

Apparently, Cases (3-1) and (3-2) (or Cases (3-3) and (3-4)) give the same prediction concerning $\pi(or\ K)-\mu-e$ chain, which is consistent with b), b_1) and d). (About these arguments, see refs. 11), 1) and 3).) The behavior of $\pi^+(or\ K^+)-\mu^+-e^+$ chain predicted by (3-1) (or (3-2)) and (3-3) (or (3-4)) where $\psi_\nu'=\psi_\nu$ in the latter case, are schematically illustrated in Fig. 2, Case $II_\mu b$ (see also Fig. 1 in ref. 12)), and Fig. 4, Case $III_\mu b$ in I respectively.

In order to explain the largeness of the asymmetry coefficient α in $\Lambda\to p+\pi^-$ decay, $(\bar{\Lambda}\ p)\ (\bar{p}\ n)$ is desired to be of V–A type.* Thus, iv) in (3-1) and (3-2) should be respectively of the form

$$f(\bar{\psi}_\Lambda(1\pm\gamma_5)\gamma_\rho\ \psi_n)\ (\bar{\psi}_p(1\pm\gamma_5)\gamma_\rho\ \psi_p)$$

and

$$f(\bar{\psi}_\Lambda(1\mp\gamma_5)\psi_n)\ (\bar{\psi}_p(1\pm\gamma_5)\psi_p).^{**}$$

But, as we can see later, the forms of iv) have nothing to do with our conclusion.

If we take S–T–P as the coupling type of β-decay of nucleons and admit the fact a), four cases that give the consistent phase relation are obtained. (See Appendix 2.) Even if we do not know the stumbling block inherent in the S–T–P coupling (which has the difficulty to account for the suppression of β-decay of bosons), we can discover that this coupling should be discarded. Because, the S–T–P coupling gives the forward emission of e^+ in π^+ (*or* K^+) $-\mu^+-e^+$ chain, which contradicts definitely with the well established fact b). (See Figs. 2 and 4 in I.) Therefore, we can assert that the β-decay interaction of nucleons is of V–A type. This is consistent with the experimental results obtained by Herrmannsfeldt, et al.[14] and Goldhaber, et al.[15], but inconsistent with the result of He^6 recoil experiment by Rustad and Ruby.[16]

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* Using the gradient boson-fermion coupling, we get the asymmetry coefficient $\alpha_{cal}(\Lambda^0\to p+\pi^-)\sim 0.9$, and on the other hand, in the case of the direct coupling, $\alpha_{cal}\sim -0.1$[4)13)]

** Assuming $\sum_{i=1}^{5} f(\bar{\psi}_\Lambda(1\pm\gamma_5)O_i\psi_n)\ (\bar{\psi}_n\ O_i'\ \psi_n)$ to have the same form as iv), we can get the required forms of iv) in (3–1) and (3–2).

One of the differences among (3-1) to (3-4) is found in ρ value, i.e., for (3-1) and (3-2) $\rho=3/4$, and, for (3-3) and (3-4) where $\psi_\nu'=\psi_\nu$, $\rho=(9/16)(1-\alpha)$, where α is the asymmetry coefficient for μ decay.[17] The small discrepancy of the experimental value of ρ[17] from 3/4 in naturally explained by the latter.[1] However, it may be said that (3-1) and (3-2) are more preferable than (3-3) and (3-4) because of the following reasons. First, the universality of the Fermi interaction containing leptons is realized in (3-1) and (3-2). Secondly, if we assign the same lepton number to μ^- and e^+ (which means $\psi_\nu'=\psi_\nu^c$) in order to forbid many unwanted processes such as $\mu^\pm\to e^\pm+e^++e^-$, $\mu^\pm\to e^\pm+\gamma$ and $\mu^\pm+\mathcal{N}\to e^\pm+\mathcal{N}$, etc., then (3-3) and (3-4) are definitely excluded, because $\rho=0$. This assignment, combined with the assumption IV), implies that all leptonic processes must be accompanied with neutrino.* The experiment to measure the polarization of μ^- emitted by π^- or K^- and the investigation of μ^--capture interaction[19] are important to check the validity of the universal V–A coupling.

§ 4. Concluding remarks

From our discussion it was concluded phenomenologically that the type of β-decay interaction must be of V–A and that the universal V–A coupling of the Fermi interaction containing leptons is a natural consequence. Our result is consistent with the assumption made by Feynman and Gell-Mann,[4] Sakurai[4] and Marshak and Sudershan[4], and the analysis by H. Umezawa, et al.[18] According to our standpoint, the gradient weak boson-fermion interaction should be an apparent form of the V–A type Fermi interaction.

Assigning the same lepton number to μ^- and e^- and admitting our assumptions, we are led to conclude that leptonic processes are always accompanied with neutrino. Therefore we can get the same selection rule as pointed out by Feynman and Gell-Mann[4] and Ōneda:[20] $\Delta Q(\textit{baryonic charge})=+1$ when $\Delta S=+1$.

We are interested in the process dependency of effective coupling constants when Fermi interactions are reduced to effective boson-fermion interactions. The A-factor in (2-1) for various decay processes which really occur should be of the same order ($10^{-1}\sim10^{-7}$). Observing available decay processes, we can find some tendency that the larger the Q-value is, the smaller the decay probability becomes. This Q-value dependence seems to us to be the character of A-factor in (2−1).

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Appendix 1

In (3-1) to (3-4), if $\psi_\nu\to\eta_\nu^\nu\gamma_5\psi_\nu$, then $\psi_\nu^c\to-\eta_\nu^{\nu*}\gamma_5\psi_\nu^c\equiv\eta_\nu^{\nu\prime}\gamma_5\psi_\nu^c$. The expressions of (3-1) are obtained as follows. Accepting i) in (3-1),

* While, if we assign the same lepton number to μ^- and e^-, only for (3−1) and (3−2) we are allowed to assign the lepton number to neutrinos in the sense of the two component theory.[18]

$$\left.\begin{aligned}&\eta(a: pne\nu') \equiv \eta_p^{a*}\,\eta_n^a\,\eta_e^{a*}\,\eta_\nu^a = \mp 1 \text{ for } a=p \text{ and } n,\\ &\eta(a: pne\nu') = +1 \text{ for } a=e,\ \nu,\ \mu \text{ and } \Lambda.\end{aligned}\right\} \qquad \text{(A–1)}$$

As μ-capture and μ-decay really occurs,

$$\eta(a: pn\mu\nu) = +1 \text{ for } a=e \text{ and } \Lambda, \qquad \text{(A–2)}$$

and

$$\eta(a: e\mu\nu\nu') = +1 \text{ for } a=n,\ p \text{ and } \Lambda. \qquad \text{(A–3)}$$

From (A–1) and (A–3), it follows

$$\eta(a: pn\mu\nu) = \mp 1 \text{ for } a=p \text{ and } n. \qquad \text{(A–4)}$$

Therefore, irrespective of the type of μ-decay interaction, the μ-capture interaction is of V–A and the double signs must be of the same order as i) in (3–1). From (A–1) and (A–2), it follows

$$\eta(a: e\mu\nu\nu') = +1 \text{ for } a=e \text{ and } \Lambda. \qquad \text{(A–5)}$$

Assuming that the type of μ-decay interaction is of V–A, we get the expression ii) in (3–1). Then,

$$\eta(a: e\mu\nu\nu') = +1 \text{ for all } a\text{'s}. \qquad \text{(A–6)}$$

From (A–1) and (A–6), the following is deduced beside (A–2) and (A–4),

$$\eta(a: pn\mu\nu) = +1 \text{ for } a=\nu \text{ and } \mu.$$

Then iii) in (3–1) must be chosen. In the same way, we can get iv) to vi) in (3–1).

Appendix 2

If S-T-P coupling is taken as the β-decay or μ-capture interaction of nucleons, the following cases are possible :

$$\begin{aligned}&\text{i)} && \sum_{S,T} f(\bar{\psi}_p(1-\gamma_5)\,o\,\psi_n)\,(\bar{\psi}_e(1-\gamma_5)\,o\,\psi_\nu'),\\ &\text{ii)} && \sum_{S,T,P} f(\bar{\psi}_e(1-\gamma_5)\,o\,\psi_\mu)\,(\bar{\psi}_\nu\,o\,\psi_\nu'),\\ &\text{iii)} && f\,(\bar{\psi}_p(1-\gamma_5)\psi_n)\,(\bar{\psi}_\mu(1+\gamma_5)\psi_\nu),\\ &\text{iv)} && \sum_{S,T,P} f(\bar{\psi}_\Lambda(1-\gamma_5)\,o\,\psi_n)\,(\bar{\psi}_p\,o\,\psi_p),\\ &\text{v)} && f(\bar{\psi}_p(1+\gamma_5)\psi_\Lambda)\,(\bar{\psi}_e(1-\gamma_5)\psi_\nu'),\\ &\text{vi)} && \sum_{S,T} f(\bar{\psi}_p(1+\gamma_5)\,o\,\psi_\Lambda)\,(\bar{\psi}_\mu(1+\gamma_5)\,o\,\psi_\nu),\end{aligned} \qquad \text{(A–7)}$$

i), ii) and iii) are as same as those in (A–7), and

$$\begin{aligned}&\text{iv)} && f(\bar{\psi}_\Lambda(1+\gamma_5)\gamma_p\psi_n)\,(\bar{\psi}_p\,\gamma_p'\,\psi_p),\\ &\text{v)} && \sum_{S,T} f(\bar{\psi}_p(1-\gamma_5)\,o\,\psi_\Lambda)\,(\bar{\psi}_e(1-\gamma_5)\,o\,\psi_\nu'),\\ &\text{vi)} && f(\bar{\psi}_p(1-\gamma_5)\psi_\Lambda)\,(\bar{\psi}_\mu(1+\gamma_5)\psi_\nu),\end{aligned} \qquad \text{(A–8)}$$

i), iv) and v) are as same as those in (A–7), and

ii) $f(\bar{\psi}_e(1-\gamma_5)\gamma_\rho\,\psi_\mu)\,(\bar{\psi}_\nu\,\gamma_\rho'\,\psi_\nu')$,

iii) $\sum_{S,T} f(\bar{\psi}_p(1-\gamma_5)o\,\psi_n)\,(\bar{\psi}_\mu(1-\gamma_5)o\,\psi_\nu)$, (A–9)

vi) $f(\bar{\psi}_p(1+\gamma_5)\psi_\Lambda)\,(\bar{\psi}_\mu(1-\gamma_5)\;\psi_\nu)$,

and

i), ii) and iii) are as same as those in (A–9) and

iv) and v) as same as those in (A–8), and (A–10)

vi) $\sum_{S,T} f(\bar{\psi}_p(1-\gamma_5)o\,\psi_\Lambda)\,(\bar{\psi}_\mu(1-\gamma_5)o\,\psi_\nu)$.

The reason for that $(1+\gamma_5)$ disappears from the nucleon part of β-decay interaction is that the tensor coupling is necessary as far as the allowed Gamow-Teller transition in β-decay occurs.

Any choice from (A–7) to (A–10) gives inconsistent results with experiment. Because, if the spiralities of e^- and μ^- emitted respectively from μ^- and π^- (or K^-) are determined according to the corresponding interaction Hamiltonians, we see that e^- from μ^+ mesons should be emitted preferentially forward with respect to the direction of motion of μ^+ mesons.[1)12)]

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Charge-Transfer Forces in Molecular Compounds

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Charge-transfer energies or electron exchange energies are calculated between conjugated chain molecules and between benzene molecules which are formed into molecular compounds from the same species and called self-complexes. The essential nature of these energies is that these are largest in the displaced position that one molecule is laterally translated by some bond lengths to the other, and values of which cannot be negligible relative to the dispersion energies at the normal separation distance at which the molecular compounds are formed. The results obtained explain the observed phenomena qualitatively.

§ 1. Introduction

It is well known that the dispersion forces are most important and dominant among the inter-molecular forces. The dispersion energies between large chain molecules have been calculated by Coulson and Davies[1] with the method of LCAO MO (molecular orbital constructed by a linear combination of atomic orbitals) or by Haugh and Hirschfelder[2] with the Free Electron method, which are both oriented from London's treatment[3] of dipole-dipole interaction. They showed not only the long range character of the dispersion forces but also certain interesting directional properties of these inter-molecular fields of forces. However, at the distance where large chain molecules form the crystal, the charge-transfer forces may become effective. Mulliken[4] has already suggested the importance of this force in electron donor-accepter system and explained the charge transfer spectra of many molecular compounds. Also he pointed the laterally displaced systems observed in benzene crystal or graphite are energetically stable from the view-points of the symmetry natures of electronic wave functions. Recently Suzuki,[5] considering the charge-transfer forces, treated Quinhydrone which is the molecular compound between Quinone and Hydroquinone, and obtained the reasonable stabilization energy of this complex, with the use of the estimated resonance integral from the charge-transfer spectra.

I shall show that the charge-transfer forces between conjugated chain molecules and those between benzene molecules give some part of inter-molecular forces at rather short distance, and there occurs the lateral translation of one molecule relative to the other, when molecular compounds are formed. But in the following treatment the limitations are that the same kind of two molecules approach each other in the manner that the two molecular planes are parallel and the overlaps between the atomic orbitals of distinct molecules are as large as possible.

§ 2. General formulation

The ground state N of any molecular compound AB is expressed as

$$\Psi_N = a\Psi_0 + \Sigma_i b_i \Psi_i. \tag{1}$$

In eq. (1), Ψ_0 is a no-bond wave function and has the form

$$\Psi_0 = \Psi_0(\mathrm{A})\,\Psi_0(\mathrm{B}), \tag{2}$$

where $\Psi_0(\mathrm{A})$ and $\Psi_0(\mathrm{B})$ are the ground state wave functions of A and B molecule respectively. Ψ_i is a dative wave function corresponding to the transfer of one or more electrons from B to A, or A to B.

$$\Psi_i = \Psi_i(\mathrm{A}^+ - \mathrm{B}^-) + \Psi_i(\mathrm{A}^- - \mathrm{B}^+) \tag{3}$$

The coefficients a and b_i are related neglecting overlap integrals,

$$a^2 + \Sigma_i b_i^2 = 1. \tag{4}$$

Considering energy relations, the second order perturbation theory will give an adequate approximation for the loose complex,

$$E_N = \int \Psi_N H \Psi_N d\tau \fallingdotseq E_0 - \Sigma_i \frac{H_{0i}^2}{E_i - E_0} \tag{5}$$

where

$$E_0 = \int \Psi_0 H \Psi_0 d\tau, \quad E_i = \int \Psi_i H \Psi_i d\tau,$$

$$H_{0i} = \int \Psi_0 H \Psi_i d\tau.$$

H is the Hamiltonian operator for the entire set of nuclei and electrons. E_0 is equal to the sum of the separated energies of A and B in the ground state, and E_i has a similar meaning in the charge transfer state. The stabilization energy owing to the charge-transfer is

$$\Delta E = -\Sigma_i \frac{H_{0i}^2}{E_i - E_0}. \tag{6}$$

An essential requirement in eq. (1) is that Ψ_i shall be of the same group species as Ψ_0, otherwise H_{0i} is zero and there is no resonance. This means usually that Ψ_i must be of the same spin type as Ψ_0, and of the same orbital species to the overall symmetry of the complex as a whole, under the group-theoretical classification corresponding to the complex. Now suppose that $\Psi_0(\mathrm{A})$ and $\Psi_0(\mathrm{B})$ in eq. (2) are both totally symmetrical singlet state and Ψ_0 and Ψ_i must be of this same type.

In the following LCAO MO approximation is used.

§ 3. Between conjugated chain molecules

For the conjugated chain molecules which have $2m$ carbon atoms, Coulson[9] has derived the general formula: the molecular orbitals

$$\phi_n = \sum_{\lambda=1}^{2m} c_{n\lambda}\chi_\lambda \tag{7}$$

where

$$c_{n\lambda} = \sqrt{\frac{2}{2m+1}} \sin\frac{n\lambda\pi}{2m+1}$$

and corresponding energies, relative to the energy of an isolated $2p\pi$ carbon atomic orbital χ, are

$$\varepsilon_n = 2\beta_\pi \cos\frac{n\pi}{2m+1} \tag{8}$$

where β_π, which is negative, is the resonance integral between adjacent atoms. In these formula, the overlap integrals are neglected, between distinct atomic orbitals. Such neglect is known to make very little difference to the energy in the ground state.

The ground state of each molecule is obtained by filling each of the lower m orbitals with a pair of electrons according to the Pauli principle. The charge-transfer states are given by configurations in which one or more electrons are translated from the lower bonding orbitals of one molecule to the upper anti-bonding orbitals of the other molecule. From these configurations only the most effective ones are selected, under the symmetry consideration, that are shown in Fig. 1.

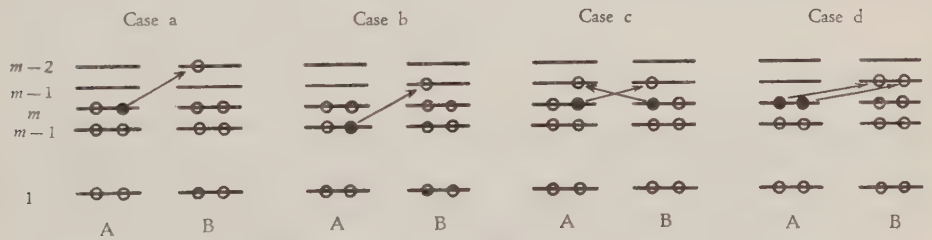


Fig. 1. Charge-transfer states between conjugated chain molecules.

Only Case a and Case b shall be considered, because Case c and Case d give only negligibly small contributions, proportional to the square of the overlap integral.

Then the stabilization energy of the complex, relative to the separated molecules is

$$\begin{aligned} \Delta E &= \Sigma_i \frac{H_{0i}^2}{E_i - E_0} \\ &= 2\left\{\frac{H'(\mathrm{a})^2}{\varepsilon_{m+2}(\mathrm{B}) - \varepsilon_m(\mathrm{A})} + \frac{H'(\mathrm{b})^2}{\varepsilon_{m+1}(\mathrm{B}) - \varepsilon_{m-1}(\mathrm{A})}\right\} \\ &\quad + 2\left\{\frac{H'(\mathrm{a})^2}{\varepsilon_{m+2}(\mathrm{A}) - \varepsilon_m(\mathrm{B})} + \frac{H'(\mathrm{b})^2}{\varepsilon_{m+1}(\mathrm{A}) - \varepsilon_{m-1}(\mathrm{B})}\right\}, \end{aligned} \tag{9}$$

where $H'(\mathrm{a})$ and $H'(\mathrm{b})$ are interaction energies of Case a and Case b respectively. The denominators of eq. (9)

$$\varepsilon_{m+2}-\varepsilon_m=\varepsilon_{m+1}-\varepsilon_{m-1}$$

$$=-2\beta_\pi\left\{\cos\frac{m}{2m+1}\pi+\cos\frac{m-1}{2m+1}\pi\right\} \tag{10}$$

are easily verified. Next $H'(\mathrm{a})$ and $H'(\mathrm{b})$ must be expressed in terms of atomic orbitals and their coefficients in each molecular orbitals. Now suppose, as in Fig. 2, that the two molecules A and B approach in such a manner that their molecular planes are pararell with each other, and then the interactions between electrons of π-orbitals are σ-type. It is obvious that such an approach obtains the greatest interaction energies. Here, by example, the explicit expressions of $H'(\mathrm{a})$ and $H'(\mathrm{b})$ corresponding to Fig. 2 are given

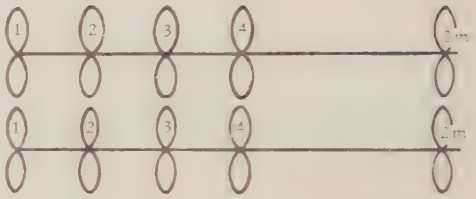


Fig. 2. An approach of two conjugated chain molecules.

$$H'(\mathrm{a})=\int\sum_{\lambda=1}^{2m}c_{m,\lambda}(\mathrm{A})\chi_\lambda(\mathrm{A})H'\sum_{\mu=1}^{2m}c_{m+2,\mu}(\mathrm{B})\chi_\mu(\mathrm{B})\,dv$$

$$=-\frac{1}{2}A_{2m,0}(\mathrm{a})[0]+A_{2m,1}(\mathrm{a})[1]+\cdots\cdots+A_{2m,2m-1}(\mathrm{a})[2m-1], \tag{11}$$

where H' is the perturbing potential.

$$[\kappa]=\int\chi_\lambda(\mathrm{A})H'\chi_{\lambda-\kappa}(\mathrm{B})\,dv \tag{12}$$

and

$$A_{2m,\kappa}(\mathrm{a})=\sum_{\mu=1}^{2m-\kappa}\left\{c_{m,\mu}(\mathrm{A})\,c_{m+2,\mu+\kappa}(\mathrm{B})+c_{m,\mu-\kappa}(\mathrm{A})\,c_{m+2,\mu}(\mathrm{B})\right\}$$

$$=\sum_{\mu=1}^{2m+\kappa}{}'\left\{\sin\frac{m\mu\pi}{2m+1}\sin\frac{(m+2)(\mu-\kappa)}{2m+1}\pi+\sin\frac{m(\mu-\kappa)}{2m+1}\pi\sin\frac{(m+2)\mu}{2m+1}\pi\right.$$

$$=(-1)^{\kappa+1}\left\{\cos\frac{m\kappa}{2m+1}\pi\cdot\sin\frac{\kappa+1}{2m+1}\pi\Big/\sin\frac{1}{2m+1}\pi\right.$$

$$\left.-\cos\frac{\kappa}{2m+1}\pi\cdot\sin\frac{m(\kappa+1)}{2m+1}\pi\Big/\sin\frac{m}{2m+1}\pi\right\}. \tag{13}$$

The similar equation for $H'(\mathrm{b})$ is obtained

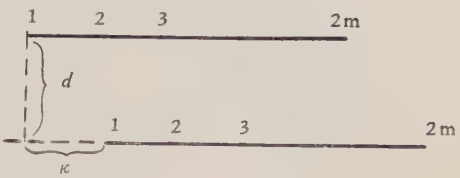
$$H'(\mathrm{b})=\frac{1}{2}A_{2m,0}(\mathrm{b})[0]+A_{2m,1}(\mathrm{b})[1]+\cdots\cdots+A_{2m,2m-1}(\mathrm{b})[2m-1] \tag{14}$$

but

$$A_{2m,\kappa}(\mathrm{b})=-\left\{\cos\frac{m\kappa}{2m+1}\pi\cdot\sin\frac{\kappa+1}{2m+1}\pi\Big/\sin\frac{1}{2m+1}\pi\right.$$

$$\left.-\cos\frac{\kappa}{2m+1}\pi\cdot\sin\frac{m(\kappa+1)}{2m+1}\pi\Big/\sin\frac{m}{2m+1}\pi\right\}. \tag{15}$$

The charge-transfer energies between the chain molecules of the same kind will depend on the size of molecules and on their mutual coordinates. The size of molecules is simply determined by $2m$, the numbers of carbon atoms. For the purpose of numerical evaluation, the interatomic distance within each molecule is taken as 1.4 A, which is the value appropriate to a conjugated carbon chain. The mutual coordinates used are shown in Fig. 3.



d: normal separation of chains.
κ: lateral translation of one chain relative to the other measured by the unit of one bond length.

Fig. 3. Mutual coordinates between two molecules.

The resonance integral $[\kappa]$ is separated into σ-type and π-type ones,

$$[\kappa]=\cos^2\theta[\kappa]_\pi+\sin^2\theta[\kappa]_\sigma. \tag{16}$$

$[\kappa]_\pi$ and $[\kappa]_\sigma$ are estimated in unit of β_π and β_σ, which are respectively resonance integrals of $2p\pi$ and $2p\sigma$ atomic orbitals at the distance 1.4 A in the field of molecular chain. Mulliken estimated β_π as -40 kcal/mol from the spectroscopic data, while β_σ is here roughly supposed to be -100 kcal mol, considering this value to be proportional to the dissociation energy of the single bond between carbon atoms. Moreover it is assumed that these values depend upon the distance in the same manner as overlap integrals between corresponding atomic orbitals do.

As an example, butadiene shall be considered.

(1) $\kappa=0$

$$H'=\tfrac{1}{2}A_{1,0}[0]+A_{1,0}[1]+A_{1,2}[2]+A_{3,3}[3].$$

The same formula stands for a and b.

(2) $\kappa=1$

$$H'=\tfrac{1}{2}A_{1,1}[0]+\tfrac{1}{2}(A_{1,0}+A_{1,2})[1]+\tfrac{1}{2}(A_{1,1}+A_{3,3})[2]+\tfrac{1}{2}A_{1,2}[3]+\tfrac{1}{2}A_{3,3}[4]$$

(3) $\kappa=2$

$$H'=\tfrac{1}{2}A_{1,2}[0]+\tfrac{1}{2}(A_{1,1}+A_{3,3})[1]+\tfrac{1}{2}A_{4,0}[2]+\tfrac{1}{2}A_{3,1}[3]+\tfrac{1}{2}A_{4,2}[4]+\tfrac{1}{2}A_{4,2}[5]$$

(4) $\kappa=3$

$$H'=\tfrac{1}{2}(A_{4,3}[0]+A_{4,2}[1]+A_{4,1}[2]+A_{4,0}[3]+A_{4,1}[4]+A_{4,2}[5]+A_{4,3}[6])$$

In practical calculations, the smaller terms than $|4|$ are neglected. For larger molecules, similar equations are easily obtained. Results of calculation are tabulated in the following.

From the foregoing calculation, we can know (1) the charge-transfer energy exceedingly decreases as normal separation of molecular planes increases. That is, the dispersion energy is the function of the reciprocal power of distance, but the charge-transfer energy is the function of the exponent of distance. Therefore the charge-transfer force is notable at rather short distance, where the valence force is about to play its part. (2)

Table 1. $A_{2m,\kappa}$ (a) : upper, $A_{2m,\kappa}$ (b) : lower

κ \ $2m$	4	6	8	10
0	0 0	0 0	0 0	0 0
1	0 0	0 0	0 0	0 0
2	1.1186 1.1186	1.5267 1.5267	1.7060 1.7060	1.7993 1.7993
3	−1.1186 1.1186	−1.2240 1.2240	−1.1140 1.1140	−0.9851 0.9851
4		−1.2240 −1.2240	−2.0913 −2.0913	−2.6354 2.6354
5		1.5267 −1.5267	2.0913 −2.0913	2.1879 −2.1879
6			1.1140 1.1140	2.1879 2.1879
7			−1.7060 1.7060	−2.6354 2.6354
8				−0.9851 −0.9851
9				1.7993 −1.7993

Table 2. Charge-transfer energies between conjugated chain molecules (kcal/mol)

$2m$	κ	2 A	4 A	8 A	15 A
4	0	1.25	0.130^{-1}	0.593^{-9}	0.149^{-24}
	1	29.7	0.266^{-1}	0.138^{-8}	0.181^{-24}
	2		0.308	0.307^{-8}	0.252^{-24}
	3		0.308	0.304^{-8}	0.235^{-24}
	*		0.935	0.341^{-1}	0.106^{-2}
6	0	3.06	0.274^{-1}	0.138^{-8}	0.292^{-24}
	1	72.2	0.181	0.289^{-8}	0.296^{-24}
	2		0.594	0.445^{-8}	0.234^{-24}
	3		0.288	0.164^{-8}	0.911^{-25}
	4		0.287	0.154^{-8}	0.467^{-25}
	5		0.593	0.439^{-8}	0.208^{-24}
	*		2.62	0.140	0.558^{-2}
8	4		1.06		
10	4		1.99		
	*		7.95		

$0.130^{-1}=0.130\times10^{-1}$

*: Dispersion energies obtained by Coulson and Davies.

The charge-transfer force is most effective at the definite lateral translation of one molecule to the other, which is characteristic of the sort of the molecule. Such circumstance is known from $A_{2m,\kappa}$ values of Table 1, that is to say, the charge-transfer force is strongest at the relative position where the absolute value of $A_{2m,\kappa}$ is highest. However, other factors, e. g. the steric hindrance and the repulsions of attached hydrogens, etc., are considered together.

§ 4. Between benzene molecules

It is known that benzene molecules are constructed into a crystalline state as shown in Fig. 4. In Fig. 4 (a) the full line and dotted line both show benzene molecular planes which are perpendicular to paper, and the former is translated by about a half width of molecular plane relatively to the latter. The similar structure is observed in graphite. A pair of A and B molecules in the frame of Fig. 4 (a) is taken out as a unit and will be discussed, which is looked down as Fig. 4 (b) generally and at the case of graphite as Fig. 4 (c). Coulson[7] has pointed out: If the dispersion force is dominant among the inter-molecular forces between two benzene molecules, the less-directional nature of this force makes it possible for benzene molecules to pack together as closely as possible. And any molecule (say the full lined one of Fig. 4 (a)) is surrounded by twelve neighbours, four full lined ones at the same horizontal level, and four dotted lined ones above and below. This illustration is satisfactory, but cannot deny

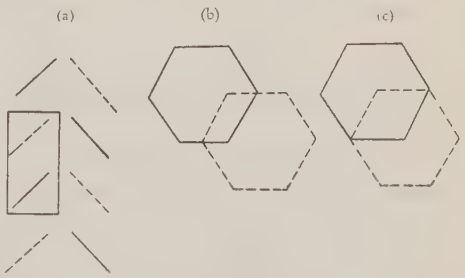


Fig. 4. Crystalline state of benzene molecules.

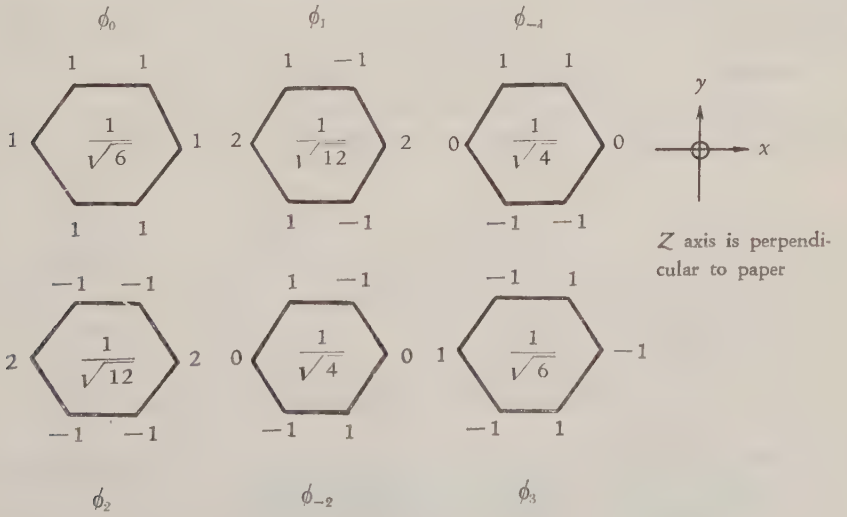


Fig. 5. Molecular orbitals of benzene.

other sources for the structure of benzene crystal. The charge-transfer forces between benzene molecules support this structure energetically, which is not expected from the view-point of the dispersion forces.

The molecular orbitals of benzene obtained by the linear combination of six $2p\pi$ carbon atomic orbitals are shown in Fig. 5. The numbers on vertexes of the hexagon are the coefficients of linear combination respectively to each $2p\pi$ carbon atomic orbital, and the numbers inside are normalization factors, where, needless to say, overlap integrals are neglected. The symmetry natures of these orbitals are tabulated in Table 3 referring to the coordinate system in Fig. 5. $\sigma(x)$ is reflection in the xz plane and $\sigma(y)$ is reflection in the yz plane, and i is inversion in the center of symmetry, but the signs

Table 3. Symmetry natures of benzene molecular orbitals

	ϕ_0	ϕ_1	ϕ_{-1}	ϕ_2	ϕ_{-2}	ϕ_3
$\sigma(x)$	+	+	−	+	−	+
$\sigma(y)$	+	−	+	+	−	−
i	+	−	−	+	+	−

in i are taken reversely, considering A−B complex shall be formed. A and B benzene molecules in the non-bonded state of eq. (2) fill ϕ_0 and ϕ_1 orbitals with two electrons respectively, according to the Pauli principle. In the charge-transfer state of eq. (3), one or more electrons in the occupied orbitals of one molecule translate into the vacant orbitals of the other molecules. As the case of the chain molecule in which only the one electron translation is considered, the relative position of A and B molecules is restricted under the symmetry natures of molecular orbitals. That is, only Case (d) in Fig. 6 corresponding to Fig. 4 (b) is allowed, and others are forbidden. For example, considering the charge-transfer from A to B in Case (b) in Fig. 6, the electron donating orbital of A and the electron accepting orbital of B share the same nature in $\sigma(x)$, but are different in i, therefore the both orbitals cannot be combined and the charge-transfer is forbidden. Only in Case (d) in Fig. 6 in which the complex has the symmetry of i only, the above mentioned two orbitals keep the same nature of symmetry i,

Fig. 6. Relative positions between A and B benzene molecules.

and the charge-transfer is allowed. The charge-transfer states corresponding to $\Psi(A^+ - B^-)$ are shown in Fig. 7, under symmetry consideration.

Then the stabilization energies of complex, relative to the separated molecules are

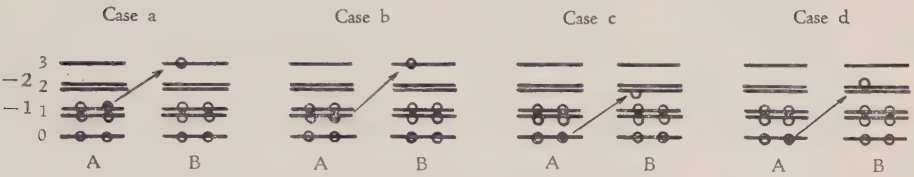
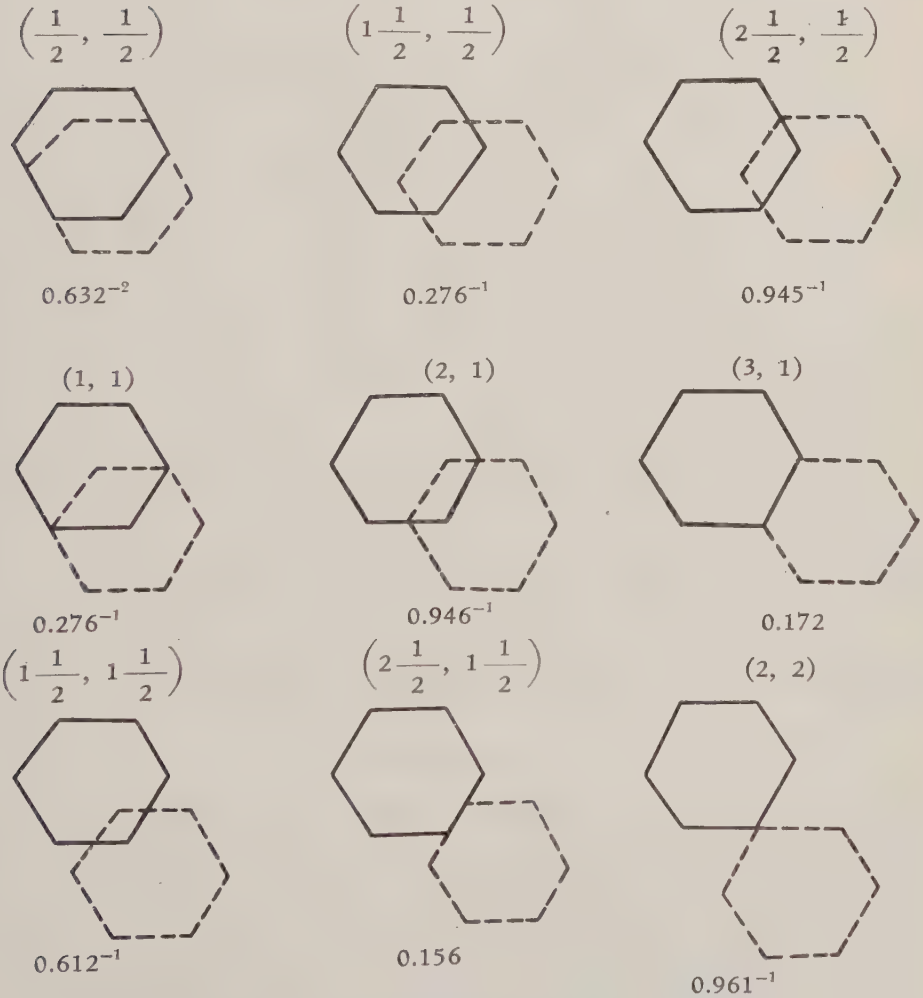


Fig. 7. Charge transfer states corresponding to $\Psi(A^+-B^-)$ between benzene molecules.

Table 4. Charge transfer energies between benzene molecules (kcal/mol)



$0.632^{-2}=0.632\times10^{-2}$ (kcal/mol)

(i, j) i: lateral translation along x axis in unit of 1/2 bond length

j: lateral translation along y axis in unit of $\sqrt{3}/2$ bond length

$$\Delta E = 2\left\{\frac{H'(a)^2}{\varepsilon_3(B)-\varepsilon_1(A)}+\frac{H'(b)^2}{\varepsilon_3(B)-\varepsilon_{-1}(A)}+\frac{H'(c)^2}{\varepsilon_2(B)-\varepsilon_1(A)}+\frac{H'(d)^2}{\varepsilon_{-2}(B)-\varepsilon_1(A)}\right.$$
$$\left.+\frac{H'(a)^2}{\varepsilon_3(A)-\varepsilon_1(B)}+\frac{H'(b)^2}{\varepsilon_3(A)-\varepsilon_{-1}(B)}+\frac{H'(c)^2}{\varepsilon_2(A)-\varepsilon_1(B)}+\frac{H'(d)^2}{\varepsilon_{-2}(A)-\varepsilon_1(B)}\right\}. \tag{16}$$

The denominators in eq. (16) are all equal to $-2{}_1\beta_\pi$. Numerical calculation is performed quite similar to the case of the chain molecule, but the normal separation between two molecules is 3.5 A (3.41 A in graphite). The results obtained are shown in Table 4.

This calculation shows that the charge-transfer energy is largest at the relative position where two benzene molecules are looked down as naphthalene molecule. Though the absolute values of the charge-transfer energies will change corresponding to the values of ${}_1\beta_\pi$ and ${}_1\beta_\sigma$, it is hardly considered that their relative values will vary. Therefore the above mentioned Mulliken's speculation corresponding to the displaced structure of benzene crystal or observed in graphite, is not fully satisfied by this calculation. While the dispersion energy between two benzene molecules was estimated as 1.50 kcal/mol by Hirschfelder with the use of Free Electron method, which is the largest value at the position where two benzene molecules are situated quite face to face as Case (a) in Fig. 6. This value is about ten times the largest value of the charge-transfer energy, so the charge-transfer energies give small contribution to the inter-molecular forces between benzene molecules. The dispersion forces are essentially caused by the intra-molecular phenomena that the dipoles associated with the electronic excitations in one molecule interact with the ones of another molecule. On the other hand, the charge-transfer forces are oriented from the inter-molecular phenomena, charge transfer or electron exchange between two molecules. It is naturally assumed that these two kinds of forces can contribute additively to the formation of molecular compounds, because they are characterized by perfectly different electronic behaviours. From this assumption one can explain qualitatively the observed displacement in benzene crystal, or in graphite as the energetically stable structure, that is the intermediate structure between the perfectly overlapped one and the naphthalene type one, when two molecules are looked down.

§ 5. Conclusion

The above treatment is not necessarily satisfactory on the following point: the use of semi-empirical orbitals neglecting overlap integrals and the wave functions with single product of molecular orbitals, rough estimation of ${}_1\beta_\pi$ and ${}_1\beta_\sigma$, or quite neglecting the effects from σ orbitals. Moreover, the zigzag structures of chain molecules are simplified into the linear ones, and the repulsive forces by attached hydrogens are not considered. Nevertheless the interesting results are obtained, the lateral translation between two molecules is quite essential in the charge-transfer state, which is in good agreement with the observed phenomena, and the charge-transfer energies are certainly effective at the distance of molecular compounds to be formed. It has been usual that if the shape of

molecule is large and very far from being spherical as seen in benzene, the packing is considered simply geometrical as close-packing, quite similar to the case of the metallic crystals or the case of the interaction between the rare gases. Now one may believe that the laterally displaced structures observed in benzene crystal, or in graphite, are energetically stable ones, considering the charge-transfer forces.

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Isotopic Spin Selection Rule in Σ-Decays

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Use is made of the branching ratio, lifetimes, and asymmetries of the angular distributions in Σ-decays to show that the selection rule $|\Delta I|=\frac{1}{2}$ is consistent in the decays of Σ-hyperons as well as in the Λ-decays. The invariance under time reversal is assumed, and the asymmetry parameters, the average degrees of polarization of Σ and the ratios of the coupling constants are obtained.

§ 1. Introduction

According to the generally accepted assignment of the isotopic spins to the hyperons and θ-mesons, the decays of these particles always accompany the change $\Delta I = \frac{1}{2}$ in the isotopic spin component. The change ΔI in the total isotopic spin has been discussed by many authors[1] on the basis of parity conservation or non-conservation with the assumption of time reversal invariance (T-invariance). The recent experiments on the angular asymmetries in Λ-[2] and Σ-decays seem to indicate violations of parity conservation and charge conjugation invariance[4] in the hyperon decays, whereas no evidence of a violation of T-invariance has so far been presented.

In the Λ- and θ-decays the observed ratios of the decay probabilities,

$$w(\Lambda\to n+\pi^0)/w(\Lambda)=0.32\pm0.05,^{5)}$$

$$w(\theta^+\to\pi^++\pi^0)/w(\theta_1{}^0)=(1.80\pm0.23)\times10^{-3},^{6)}$$

and

$$w(\theta_1{}^0\to2\pi^0)/w(\theta_1{}^0)=0.14\pm0.06^{5)},$$

suggest that the selection rule $\Delta I = \frac{1}{2}$ is dominant in these decays, while the contributions from $|\Delta I|=\frac{3}{2}$ and $\frac{5}{2}$ are about 10^{-2} of the former in the case of θ-decays.[3]

In this paper we emphasize that the analysis of the observed asymmetries together with the branching ratio and the lifetimes of the various modes of Σ-decays, can serve to determine the kinematics of the decays and the polarization of Σ-hyperons before the decay. We are thus in a position to examine the ΔI selection rules to some extent.

§ 2. Derivation of asymmetry parameters

Let the s- and p-wave amplitudes of the final nucleon pion systems be (a_-, b_-), (a_0, b_0), and (a_+, b_+) corresponding to the modes of decays (the spin of Σ-hyperons is assumed to be $\frac{1}{2}$):

$$\Sigma^+\to n+\pi^+,\ \Sigma^+\to p+\pi^0 \quad \text{and} \quad \Sigma^-\to n+\pi^-.$$

The asymmetry parameters for these decays are given by[7]

$$\alpha_i=2\,\mathrm{Re}\,(a_i^*b_i)/(|a_i|^2+|b_i|^2), \tag{1}$$

where

$$i=+,\ 0,\ -,$$

respectively. The simple isotopic spin consideration shows that

$$a_-=(a_++2^{1/2}a_0)/\lambda \quad \text{and} \quad b_-=(b_++2^{1/2}b_0)/\lambda,$$

with either $\lambda=1$ for $|\Delta I|=1/2$, or $\lambda=-2$ for $|\Delta I|=3/2$.*

Actually the measurement does not give α's directly but their products with the average polarization degrees $\bar{P}^+$ and $\bar{P}^-$ of Σ^+ and Σ^-:[3]

$$\begin{aligned}\bar{P}^+\alpha_+&=-(0.36\pm0.21),\\ \bar{P}^+\alpha_0&=-(0.37\pm0.19),\\ \bar{P}^-\alpha_-&=-(0.13\pm0.26).\end{aligned} \tag{2}$$

However, the ratio of the first two is independent of the unknown quantity $\bar{P}^+$, and from eqs. (1) this ratio is expressed as:

$$\bar{P}^+\alpha_0/\bar{P}^+\alpha_+\equiv s=\{\mathrm{Re}\,(a_0^*b_0)/(|a_0|^2+|b_0|^2)\}/\{\mathrm{Re}\,(a_+^*b_+)/(|a_+|^2+|b_+|^2)\}. \tag{3}$$

The branching ratio of Σ^+-decay and the ratio of the lifetime of Σ^+ to that of Σ^- are given by

$$w(\Sigma^+\to p+\pi^0)/w(\Sigma^+\to n+\pi^+)\equiv R=(|a_0|^2+|b_0|^2)/(|a_+|^2+|b_+|^2) \tag{4}$$

and

$$\begin{aligned}\tau(\Sigma^+)/\tau(\Sigma^-)\equiv r=\{&|a_+|^2+|b_+|^2+2(|a_0|^2+|b_0|^2)\\ &+2\sqrt{2}\,\mathrm{Re}\,(a_+^*a_0+b_+^*b_0)\}/\lambda^2(|a_+|^2+|b_+|^2+|a_0|^2+|b_0|^2).\end{aligned} \tag{5}$$

The observed values of s, R and r are:

$$\begin{aligned}s&=1.03\pm0.80 \qquad \text{from (2) and (3)},\\ R&=0.88\pm0.12^{8)}\\ r&=0.470\pm0.094.^{9)}\end{aligned} \tag{6}$$

The assumption of T-invariance and the neglect of the final state interaction** imply that a's and b's are all real. In this case eqs. (3), (4) and (5) determine the relative values of these amplitudes and consequently α's by eqs. (1), in terms of s, R

* The case of $|\Delta I|=5/2$ gives branching ratio R which is inconsistent with the observed value.

** More exactly, it is sufficient for the following discussions to make approximations $\cos(\delta_\alpha-\delta_\beta)\fallingdotseq 1$, but not always $\cos\delta_\alpha\fallingdotseq 1$. Here δ_α is the scattering phase shift of the pion-nucleon system in the final state $\alpha=(I;J=\frac{1}{2};I)$. These approximations are correct within the errors of about 10%.[1]

and r. The results of some manipulation show that

$$\begin{aligned}
\alpha_+ &= \pm 2\xi\{(1-\xi^2)\}^{-1}\{(1+s)^2-4s\xi^2\}^{1/2},\\
\alpha_0 &= s\alpha_+,\\
\alpha_- &= \alpha_+\{(1+2sR)-(1+s)\sqrt{2R}/\xi\}/\lambda^2 r(1+R),
\end{aligned} \tag{7}$$

where

$$\xi = \{(1+2R)-\lambda^2 r(1+R)\}/2\sqrt{2R}. \tag{8}$$

The degrees of polarization $\bar{P}^+$ and $\bar{P}^-$ can be obtained from (2) and (7). The numerical results obtained with the aid of the empirical values (6) are given in Table I. The table also lists the results with the simple set:

$$s=1,\ R=1 \quad \text{and} \quad r=\tfrac{1}{2}, \tag{9}$$

which are in conformity with the data (6) within the experimental errors.

Table I. Asymmetry parameters and average polarization degrees

	$\lvert\Delta I\rvert=1/2$		$\lvert\Delta I\rvert=3/2$	
s, R, r	empirical set (6)	simple set (9)	empirical set (6)	simple set (9)
α_+	$\pm(0.68\pm0.29)$	$\pm 1/\sqrt{2}$	$\pm(0.32\pm0.35)$	$\pm 1.2\sqrt{2}$
α_0	$\pm(0.70\pm0.28)$	$\pm 1/\sqrt{2}$	$\pm(0.33\pm0.36)$	$\pm 1/2\sqrt{2}$
α_-	$\mp(0.84\pm0.23)$	$\mp 1/\sqrt{2}$	$\mp(0.49\pm1.3)$	$\pm 11/8\sqrt{2}$
P^+	$\mp(0.53\pm0.22)$	$\mp(0.50\pm0.30)$		$\mp(1.0\pm0.6)$
P^-	$\pm(0.15\pm0.31)$	$\pm(0.18\pm0.37)$		$\mp(0.13\pm0.27)$

§ 3. Discussion

Our test of consistency is to see if $|P^\pm|<1$, as is required from the definition of the polarization degree. From the table we see that the selection rule $\Delta I=\frac{1}{2}$ is quite consistent. The results for the case of $\Delta I=3/2$ contain errors of more than 100% and nothing definite can be said. However, if we could disregard the errors, $|\alpha_+|$ and $|\alpha_0|$ are smaller than the corresponding valus of $|P\alpha|$'s, and $\Delta I=3/2$ could be ruled out.

The analysis of the decays of θ-mesons and Λ-hyperons has already shown the necessity of the dominant $|\Delta I|=\frac{1}{2}$. Our results show that the same selection rule $|\Delta I|=\frac{1}{2}$ is acceptable for the Σ-decays, and that the contribution from $\Delta I>\frac{1}{2}$ could be zero in this case as well as in Λ-decays, in contrast with the case of θ-decays where such contribution is not negligible.

Finally it is interesting to obtain the ratios of the coupling constants from the above analysis. If we assume the interaction Hamiltonian for the decay to be either

$$H_{\text{int}}=\bar{\Sigma}^{+}(g_{+}+g_{+}'\gamma_5)n\pi^{+}+\bar{\Sigma}^{+}(g_0+g_0'\gamma_5)p\pi^{0}$$
$$+\bar{\Sigma}^{-}(g_{-}+g_{-}'\gamma_5)n\pi^{-}+\text{h. c.}, \tag{10}$$

or

$$H_{\text{int}}=\bar{\Sigma}^{+}\gamma_\lambda(g_{+}+g_{+}'\gamma_5)n\partial_\lambda\pi^{+}+\bar{\Sigma}^{+}\gamma_\lambda(g_0+g_0'\gamma_5)p\partial_\lambda\pi^{0}$$
$$+\bar{\Sigma}^{-}\gamma_\lambda(g_{-}+g_{-}'\gamma_5)n\partial_\lambda\pi^{-}+\text{h. c.}, \tag{11}$$

the s- and p-wave amplitudes are directly connected to the coupling constants g's and g''s as*

$$b_i/a_i=\left[\{(M-m)^2-\mu^2\}/\{(M+m)^2-\mu^2\}\right]^{1/2}(g_i'^{*}/g_i^{*})$$

or

$$b_i/a_i=\{(M+m)/(M-m)\}$$
$$\cdot\left[\{(M-m)^2-\mu^2\}/\{(M+m)^2-\mu^2\}\right]^{1/2}(g_i'^{*}/g_i^{*}),$$

respectively, where M, m and μ are the masses of Σ, nucleon and pion, respectively.

Table II. Ratios of coupling constants

	Direct coupling (10)		Derivative coupling (11)	
g_{+}'/g_{+}	± 3.9	±25	±0.46	±3.0
g_0'/g_0	±24	± 4.1	±2.9	±0.48
g_{-}'/g_{-}	∓18	∓ 5.4	∓2.2	∓0.64

Table II gives the results of the calculation with the empirical set (6) for the case of $|\Delta I|=\frac{1}{2}$. It is to be noted that the assumption of $|\Delta I|=\frac{1}{2}$ definitely excludes the "one-to-one law,"[10] which seems valid in the weak interactions involving neutrinos.

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* The renormalization of the coupling constants due to strong interactions is not considered here.

$+\pi_0)/w(K^+)=(23.2\pm2.23)\%$, and F. Eisler et al. (Reference 5)) give $\tau(\theta_1^0)=(0.95\pm0.08)\times10^{-10}$ sec.

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Note added: After the completion of this work we were informed that discussions on Σ-decays taking their angular asymmetries into account had also been done independently by Y. Yamaguchi Prog. Theor. Phys. **19** (1958), 485) and K. Itabashi (private communication, Prog. Theor. Phys. **19** (1958), 747L)

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Electromagnetic Properties of Nucleons

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The current and the magnetic moment of the nucleon are calculated, starting from a relativistic one nucleon formalism. It is shown that the relativistic corrections are very important for the current and the magnetic moment of the nucleon, especially for the vector part of the former and the scalar part of the latter, where the effects of the recoils are of the same order as those given by the static model. The numerical result, obtained by means of Chew's method, shows that the neutron current and the scalar part of the magnetic moment may be consistent with the experiment by choosing suitably the coupling constant ($g^2/4\pi=0.08$ or 0.10) and the cut-off momentum ($k_{\max}=5\mu$, 6μ or 7μ). However the effect of the nucleon recoil to the vector part of the moment is destructive and gives a definitely smaller value than the empirical one. It is also shown that the renormalized charge is equal to the unrenormalized charge in our approximation which contains only the terms up to the order of $1/M$.

§ 1. Introduction

There are many meson theoretical approaches about the magnetic moment of the nucleon, among which the static model by Miyazawa[1] gives a fairly reliable result on the vector part of the moment. In the usual meson theory, however, it is very difficult to explain the scalar part of the moment in the static framework,[1,2] unless we introduce a strong correlation between the mesons in the cloud[2,3] or an effect of the virtual nucleon pair[4] phenomenologically. Then, there are many efforts to explain the magnetic moment by considering the effects of nucleon recoil,[3] isobar states,[5] nucleon pairs[6] and strange particles.[7] But it is found that these effects are too small to overcome the above discrepancy or rather increase it.

Since the magnetic moment, especially its scalar part, has its origin in the relativistic structure of the Dirac particle, and the effect of the nucleon recoil[3] has been studied only pure kinematically, it is significant to reinvestigate the problem in the relativistic framework. Then, we assume that the core nucleon behaves like the Dirac particle interacting with the meson field relativistically, and thus take into account the effect of the virtual nucleon pairs through the zitterbewegung of the core nucleon. The importance of the effect of the virtual pairs has been pointed out by C. Iso in his phenomenological argument.[4] The effects of the strange particles are not considered in this paper, for it may be expected to be very small.[6] Furthermore, we remove the S-wave term which appears in the Foldy transform of the γ_5-interaction. It is because the effect of this term must be suppressed considerably in the pion-nucleon scattering and that suppression cannot

be expected in the usual treatment,[9] so the careless handling of this term may produce an unreliable result.

In order to define the current and the magnetic moment unambiguously, we consider the real nucleon moving in the external electromagnetic field and separate the interaction into two parts[9),10] : one is the convection current interaction and the other is the magnetic one. Such a separation can be easily done by means of the technique developed by Suura.[11]

It is possible to renormalize the electric charge in our formalism. However, it is shown that the renormalized charge is equal to the unrenormalized charge up to the order of $1/M$, if the real neutron has no charge. This vanishing of the charge of the real neutron is the consequence of the requirement of the gauge invariance and it is satisfied only approximately in our case.

The numerical estimate can be done analogously to Miyazawa,[1] but we consider only the renormalized Born terms for the relativistic correction terms. It is shown that the relativistic effect is very important. As is seen in the final section, the nucleon current and the scalar part of the moment become consistent with experiment. This result was never obtained in the static theory. As to the vector part of the moment, the correction is destructive and reduces the value obtained by Miyazawa by about thirty per cent as a whole.

§ 2. Derivation of current and magnetic moment

Since we are interested only in the effects of the relativistic recoils, we start from the Hamiltonian

$$H=-i\alpha\nabla+\beta M+H_\pi+f_0\beta\tau\phi(x)-\frac{\beta f_0^2}{2M}\phi^2(x)-\beta\delta M. \tag{1}$$

H_π is the free Hamiltonian of the meson field, x is the coordinate of the core nucleon and ϕ, the vector in the isospin space, represents the meson field. M is the renormalized nucleon mass and f_0 is the unrenormalized coupling constant. The counter part for the mass renormalization $-\beta\delta M$ is introduced explicitly. The interaction $f_0\beta\tau\phi(x)$ involves the S-wave pair interaction besides the $\sigma\tau$ coupling term in the static limit. As is well known, this pair term produces too strong interaction to explain the S-wave scattering at low energy. Then, it would be better to remove this term completely rather than to handle it carelessly. We shall see later that the term $-(\beta f_0^2/2M)\phi^2(x)$ should be added so that our formalism agrees with Chew-Low's model in the static limit. The real nucleon with momentum $\boldsymbol{p}$ is described by the wave function $\Psi_p(x)$, which satisfies

$$\begin{aligned} &H\Psi_p(x)=\sqrt{M^2+\boldsymbol{p}^2}\,\Psi_p(x), \\ &\boldsymbol{P}\Psi_p(x)\equiv(-i\nabla+\boldsymbol{P}_\pi)\Psi_p(x)=\boldsymbol{p}\Psi_p(x), \end{aligned} \tag{2}$$

where $\boldsymbol{P}$ and $\boldsymbol{P}_\pi$ are the total and the meson field momentum, respectively.

In order to define the current and the magnetic moment of the real nucleon, we

introduce the interaction with the external electromagnetic vector potential $\boldsymbol{A}$, which is given by

$$H'=-e_0\left(\frac{1+\tau_3}{2}\right)\boldsymbol{\alpha}\boldsymbol{A}(\boldsymbol{x})+e_0\int \boldsymbol{A}(\boldsymbol{z})\,(\phi\times\nabla\phi)_3\,d\boldsymbol{z}, \tag{3}$$

where e_0 is the unrenormalized electric charge. Let us consider the matrix element $M(\boldsymbol{p}',\ \boldsymbol{p})$ of (3) between the real nucleon states with momentum $\boldsymbol{p}$ and $\boldsymbol{p}'$;

$$M(\boldsymbol{p}',\ \boldsymbol{p})=\frac{1}{(2\pi)^3}\int\langle\Psi_{p'}(\boldsymbol{x})\,|H'(\boldsymbol{x})\,|\,\Psi_p(\boldsymbol{x})\rangle\,d\boldsymbol{x}. \tag{4}$$

The requirement of the gauge invariance enables us to write $M(\boldsymbol{p}',\ \boldsymbol{p})$ in the form

$$M(\boldsymbol{p}',\ \boldsymbol{p})=-\left(\frac{e}{2M}\right)\langle u'|\left[\{\boldsymbol{A}(\boldsymbol{q})\,(\boldsymbol{p}+\boldsymbol{p}')+i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\boldsymbol{A}(\boldsymbol{q})\}\left(\frac{1+\tau_3}{2}\right)\right.$$

$$\left.+i\partial\mu\cdot\boldsymbol{q}\times\boldsymbol{A}\right]|u\rangle+\textit{higher order terms in } \boldsymbol{p} \textit{ and } \boldsymbol{p}', \tag{5}$$

where $|u\rangle$ and $|u'\rangle$ represent the bare spinors corresponding to the initial and final nucleon state, respectively. $\boldsymbol{A}(\boldsymbol{q})$ is the Fourier transform of the vector potential;

$$\boldsymbol{A}(\boldsymbol{q})=\frac{1}{(2\pi)^3}\int e^{-i\boldsymbol{q}\boldsymbol{x}}\boldsymbol{A}(\boldsymbol{x})\,d\boldsymbol{x},$$

where $\boldsymbol{q}=\boldsymbol{p}'-\boldsymbol{p}$ is the momentum transfer. In (5), the constant e is just the renormalized charge (not necessarily equal to e_0) and $\partial\mu$ is the anomalous magnetic moment of the nucleon[9,10] in unit of the nuclear magneton.

The remainder of this section is devoted to find the concrete expressions of e and $\partial\mu$. For this purpose, first we separate the motion of the center of mass in the real nucleon state with momentum $\boldsymbol{p}$. According to Suura,[11],* we can write $\Psi_p(\boldsymbol{x})$ as

$$\Psi_p(\boldsymbol{x})=\exp\{i\boldsymbol{p}\boldsymbol{X}\}\,\Psi_0(\boldsymbol{x}), \tag{6}$$

where

$$\boldsymbol{X}=\tfrac{1}{2}\{H^{-1},\ [\tfrac{1}{2}\{H,\ \boldsymbol{x}\}+\int H_\pi(\boldsymbol{z})\,(\boldsymbol{z}-\boldsymbol{x})\,d\boldsymbol{z}]\} \tag{7}$$

is the coordinate of the center of mass of the system and $\Psi_0(\boldsymbol{x})$ represents the real nucleon state at rest. The matrix element (4) can now be written as

$$M(\boldsymbol{p}',\ \boldsymbol{p})=\frac{1}{(2\pi)^3}\int\langle\Psi_0(\boldsymbol{x})\,|e^{-i\boldsymbol{p}'\boldsymbol{X}}H'(\boldsymbol{x})\,e^{i\boldsymbol{p}\boldsymbol{X}}|\,\Psi_0(\boldsymbol{x})\rangle\,d\boldsymbol{x}. \tag{8}$$

Next, we perform a unitary transformation

$$\Psi_0(\boldsymbol{x})=\exp(-i\boldsymbol{P}_\pi\boldsymbol{x})\,\Phi_0,\quad \nabla\Phi_0=0. \tag{9}$$

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* Our Hamiltonian (1) differs from Suura's by the term $-(f_0^2/2M)\beta\phi^2(\boldsymbol{x})$, but the relations (6), (7), (8) and (36) in his paper hold without any modification.

to eliminate the core nucleon coordinate. Correspondingly, any operator O is transformed according to

$$O \to \bar{O} \equiv e^{iP_\pi x} O e^{-iP_\pi x}.$$

Especially, an operator $F(z)$ of the meson field is transformed like

$$\bar{F}(z) = F(z-x). \tag{10}$$

For example, the Hamiltonian and the center of mass are transformed to

$$\bar{H} = -i\alpha\nabla + \beta M - \alpha \boldsymbol{P}_\pi + H_\pi + f_0 \rho_2 \tau\phi(0) - \frac{\beta f_0^2}{2M}\phi^2(0) - \beta\delta M, \tag{11}$$

$$\bar{\boldsymbol{X}} = \tfrac{1}{2}\{\bar{H}^{-1}, [\tfrac{1}{2}\{\bar{H}, \boldsymbol{x}\} + \int H_\pi(z) z\, dz]\}. \tag{12}$$

The Schrödinger equation is

$$\bar{H}\Phi_0 = M\Phi_0. \tag{13}$$

Applying the transformation (9) to (8) and noticing that $\exp(-i\boldsymbol{p}\bar{\boldsymbol{X}}) \cdot \exp(i\boldsymbol{p}\boldsymbol{x})$ is independent of $\boldsymbol{x}$, we perform the integration with $\boldsymbol{x}$ in (8) explicitly and obtain

$$\begin{aligned} M(\boldsymbol{p}', \boldsymbol{p}) = {} & \langle\Phi_0| \left\{ -e_0 \left(\frac{1+\tau_3}{2}\right)\alpha\boldsymbol{A} + e_0\boldsymbol{A}\int (\phi\times\Gamma\phi)_3 dz \right\} |\Phi_0\rangle \\ & + \langle\Phi_0| \frac{ie_0}{2}\boldsymbol{q}\times\boldsymbol{A}\int [\phi\times(z\times\Gamma\phi)]_3 dz' \Phi_0\rangle \\ & - i\boldsymbol{p}'\langle\Phi_0|(\bar{\boldsymbol{X}}-\boldsymbol{x}) \left\{ -e_0 \left(\frac{1+\tau_3}{2}\right)\alpha\boldsymbol{A} + e_0\boldsymbol{A}\int (\phi\times\Gamma\phi)_3 dz \right\} |\Phi_0\rangle \\ & + i\boldsymbol{p}\langle\Phi_0| \left\{ -e_0 \left(\frac{1+\tau_3}{2}\right)\alpha\boldsymbol{A} + e_0\boldsymbol{A}\int (\phi\times\Gamma\phi)_3 dz \right\} (\bar{\boldsymbol{X}}-\boldsymbol{x}) |\Phi_0\rangle. \end{aligned}$$

The further reduction can be done by means of (36) in reference 11), i. e.,

$$(\bar{\boldsymbol{X}}-\boldsymbol{x})|\Phi_0\rangle = \frac{1}{M}\left(\boldsymbol{R} - \frac{i}{2}\alpha\right)|\Phi_0\rangle, \tag{15}$$

with

$$\boldsymbol{R} = \int H_\pi(z) z\, dz. \tag{16}$$

Thus, we have

$$\begin{aligned} M(\boldsymbol{p}', \boldsymbol{p}) = {} & \langle\Phi_0| e_0 \left(\frac{1+\tau_3}{2}\right)\cdot\alpha\cdot\frac{1}{2M}\left(\boldsymbol{A}\cdot(\boldsymbol{p}+\boldsymbol{p}') - i\sigma\cdot\boldsymbol{q}\times\boldsymbol{A}\right) |\Phi_0\rangle \\ & + \langle\Phi_0| e_0\boldsymbol{A}\int (\phi\times\Gamma\phi)_3 dz\, |\Phi_0\rangle \\ & + \langle\Phi_0| \frac{ie_0}{2}\boldsymbol{q}\times\boldsymbol{A}\int [\phi\times(z\times\Gamma\phi)]_3 dz\, \Phi_0\rangle \end{aligned}$$

$$+\frac{ie_0}{M}\langle\Phi_0|\left(\frac{1+\tau_3}{2}\right)(\boldsymbol{Rq})\,(\boldsymbol{\alpha A})\,|\Phi_0\rangle$$

$$-\frac{e_0}{M}\langle\Phi_0|\int(\boldsymbol{\pi}\times\boldsymbol{A}\Gamma\phi)_3\,(\boldsymbol{z}\cdot\boldsymbol{p}+\boldsymbol{p}')\,d\boldsymbol{z}\,|\Phi_0\rangle$$

$$-\frac{ie_0}{2M}\langle\Phi_0|\left\{(\boldsymbol{Rq}),\ \int(\phi\times\boldsymbol{A}\Gamma\phi)_3 d\boldsymbol{z}\right\}_+|\Phi_0\rangle$$

$$+\frac{e_0}{2M}\langle\Phi_0|(\boldsymbol{\alpha}\cdot\boldsymbol{p}+\boldsymbol{p}')\,\boldsymbol{A}\int(\phi\times\Gamma\phi)_3 d\boldsymbol{z}\,|\Phi_0\rangle. \tag{17}$$

Now we shall eliminate the odd Dirac matrices in eq. (13). For this purpose, we apply the transformation[12]:

$$|\Phi_0\rangle=e^{-iG_1}|\phi_0\rangle\equiv\exp\left\{\frac{1}{2M_0}(i\rho_2\sigma\boldsymbol{P}_\pi+if_0\rho_1\tau\phi)\right\}|\phi_0\rangle, \tag{18}$$

where $M_0=M-\delta M$ is the unrenormalized nucleon mass. The Hamiltonian (11) is transformed according to

$$\bar{H}\to\bar{H}'=e^{iG_1}\bar{H}e^{-iG_1}=\bar{H}+[iG_1,\ \bar{H}]+\tfrac{1}{2}[iG_1,\ [iG_1,\bar{H}]]+\cdots\cdots. \tag{19}$$

The expansion is characterized by the powers of $1/M_0$. This expansion is not always justified for (f_0/M_0) is not so small in the $ps-ps$ theory, though one of the authors (A. K.) has shown that the higher order correction of $1/M_0$ is not so large to invalidate the expansion in the case of the pion nucleon scattering.[13] Then, we obtain

$$\bar{H}'=\rho_3 M+H_\pi+\rho_3\left(\frac{\boldsymbol{P}_\pi^2}{2M_0}\right)+\left(\frac{f_0}{2M_0}\right)\sigma\Gamma\tau\phi-\rho_3\delta M$$

$$+\left(\frac{f_0}{8M_0^2}\right)\{\sigma\boldsymbol{P}_\pi,\ \tau\pi\}_+ +f_0\rho_1\frac{\tau\pi}{2M_0}+\cdots\cdots. \tag{20}$$

The leading term of the dotted part is the odd term of order $(1/M_0)^2$. To eliminate the odd term $f_0\rho_1(\tau\pi/2M_0)$ from (20), we further apply the transformation:

$$|\phi_0\rangle=e^{-iG_2}|\rangle\equiv\exp\left\{\frac{-if_0}{4M_0^2}\rho_2\tau\pi\right\}|\rangle, \tag{21}$$

which does not change the even terms of (20) up to the order of $(1/M_0)^2$. Noticing that ρ_3 is equal to unity for the large components, we have

$$H|\rangle=0,$$

$$H=H_\pi+\frac{\boldsymbol{P}_\pi^2}{2M_0}+\frac{f_0}{2M_0}\sigma\Gamma\tau\phi-\frac{f_0}{8M_0^2}\{\sigma\boldsymbol{P}_\pi,\ \tau\pi\}_+-\delta M, \tag{22}$$

neglecting the terms higher than $(1/M_0)^2$. If we neglect the $(1/M_0)^2$ term in (22), it is reduced to the Hamiltonian of Chew-Low's model with the recoil kinetic energy $\boldsymbol{P}_\pi^2/2M$, where the constant δM has no significance at all, since δM causes only the

shift of the energy and the result of Chew's model is not affected by this energy shift. Applying (18) and (21) to (17) and noticing that δM is the order of zeroth power of M_0, we obtain

$$M(\boldsymbol{p}', \boldsymbol{p}) = \langle | \Big[-\Big(\frac{e_0}{2M}\Big)\Big(\frac{1+\tau_3}{2}\Big)\Big\{(\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{p}') + i\sigma\cdot\boldsymbol{q}\times\boldsymbol{A}\Big\}$$

$$+\frac{ie_0}{2}\boldsymbol{q}\times\boldsymbol{A}\int[\phi\times(z\times\Gamma\phi)]_3 dz - \frac{e_0}{M}\int(\pi\times A\Gamma\phi)_3 (z\cdot p - p')dz$$

$$-\Big(\frac{ie_0}{2M}\Big)\Big\{(\boldsymbol{Rq}),\ \int(\phi\times A\Gamma\phi)_3 dz\Big\}$$

$$-\Big(\frac{ie_0}{2M^2}\Big)\Big(\frac{1+\tau_3}{2}\Big)\Big(H_\pi\cdot\sigma\cdot\boldsymbol{q}\times\boldsymbol{A}+\{\boldsymbol{AP}_\pi,\ \boldsymbol{Rq}\}_+\Big)$$

$$+\Big(\frac{ie_0 f_0}{2M^2}\Big)(\tau\times\phi)_3(\boldsymbol{Rq})(\sigma\boldsymbol{A}) - \Big(\frac{e_0}{2M^2}\Big)(\boldsymbol{P}_\pi\cdot\boldsymbol{p}+\boldsymbol{p}')\boldsymbol{A}\int(\phi\times\Gamma\phi)_3 dz\Big|$$

$$+O\Big(\frac{1}{M^3}\Big), \tag{23}$$

where we used the fact that for the true vectors such as $\boldsymbol{P}_\pi$ and $\int(\phi\times\Gamma\phi)dz$,

$$\langle|\boldsymbol{P}_\pi|\rangle = 0, \quad \langle\Big|\int(\phi\times\Gamma\phi)dz\Big|\rangle = 0.$$

The replacement of M_0 by M does not alter the result up to the order of $(1/M)^2$ in (23).

The final expression of $M(\boldsymbol{p}', \boldsymbol{p})$ can be obtained, using the relations:

$$\Big\langle\Big|\int(\pi\times A\Gamma\phi)_3(z\cdot\boldsymbol{p}+\boldsymbol{p}')dz\Big|\Big\rangle = \frac{1}{3}(\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{p}')\Big\langle\Big|\int(\pi\times z\Gamma\phi)_3 dz\Big|\Big\rangle, \tag{24a}$$

$$\Big\langle\Big|\Big\{(\boldsymbol{Rq}),\ \int(\phi\times A\Gamma\phi)_3 dz\Big\}\Big|\Big\rangle = \frac{1}{2}\boldsymbol{q}\times\boldsymbol{A}\Big\langle\Big|(\boldsymbol{R}\times\int(\phi\times\Gamma\phi)dz + \text{h.c.})\Big|\Big\rangle, \tag{24b}$$

$$\langle|\{\boldsymbol{AP}_\pi,\ \boldsymbol{Rq}\}_+|\rangle = \frac{1}{2}\boldsymbol{q}\times\boldsymbol{A}\langle|(\boldsymbol{R}\times\boldsymbol{P}_\pi - \boldsymbol{P}_\pi\times\boldsymbol{R})|\rangle, \tag{24c}$$

$$\langle|(\tau\times\phi)_3(\boldsymbol{Rq})(\sigma\boldsymbol{A})|\rangle = \frac{1}{2}\boldsymbol{q}\times\boldsymbol{A}\langle|(\tau\times\phi)_3(\boldsymbol{R}\times\sigma)|\rangle, \tag{24d}$$

which are derived in the Appendix. Hence, we have

$$M(\boldsymbol{p}', \boldsymbol{p}) = -\Big(\frac{e_0}{2M}\Big)(\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{p}')\Big\langle\Big|\Big(\frac{1+\tau_3}{2}\Big) - \frac{2}{3}\int(\pi\times z\Gamma\phi)dz$$

$$+\Big(\frac{1}{3M}\Big)\boldsymbol{P}_\pi\cdot\int(\phi\times\Gamma\phi)dz\Big|\Big\rangle - \Big(\frac{e_0}{2M}\Big)i\boldsymbol{q}\times\boldsymbol{A}\Big\langle\Big|-M\int[\phi\times(z\times\Gamma\phi)]_3 dz$$

$$+\Big(\frac{1+\tau_3}{2}\Big)\sigma + \frac{1}{2}\Big(\boldsymbol{R}\times\int(\phi\times\Gamma\phi)dz + \text{h.c.}\Big) + \frac{H_\pi}{M}\Big(\frac{1+\tau_3}{2}\Big)\sigma$$

$$+\left(\frac{1}{2M}\right)\left(\frac{1+\tau_3}{2}\right)(\boldsymbol{R}\times\boldsymbol{P}_\pi+\text{h.c.})+\left(\frac{f_0}{2M}\right)(\boldsymbol{\tau}\times\boldsymbol{\phi})_3(\boldsymbol{\sigma}\times\boldsymbol{R})\Big\}|\rangle, \quad (25)$$

where the first term is the convection current interaction and the second the magnetic moment interaction.

§ 3. Evaluation of nucleon current and magnetic moment

Before the evaluation of the expectation values in (25), we note some gross features on each term of (25). In the following discussions it is convenient to divide both the current and the moment terms in (25) into the scalar part and the vector part in the isospin space. That is,

$$M(\boldsymbol{p}',\ \boldsymbol{p})=-\left(\frac{e_0}{2M}\right)(\boldsymbol{A}\cdot\boldsymbol{p}+\boldsymbol{p}')\langle|(Q_s+Q_v)|\rangle$$

$$-i\left(\frac{e_0}{2M}\right)\boldsymbol{q}\times\boldsymbol{A}\langle|(\mu_s+\mu_v)|\rangle, \quad (26)$$

where

$$Q_s=1/2, \quad (27a)$$

$$Q_v=\frac{\tau_3}{2}+\left(\frac{2}{3}\right)\int(\pi\times z\Gamma\phi)_3 dz+\left(\frac{1}{3M}\right)\boldsymbol{P}_\pi\cdot\int(\phi\times\Gamma\phi)_3 dz, \quad (27b)$$

$$\mu_s=\frac{\sigma}{2}+\left(\frac{H_\pi}{M}\right)\frac{\sigma}{2}+\left(\frac{1}{2M}\right)\frac{1}{2}(\boldsymbol{R}\times\boldsymbol{P}_\pi-\boldsymbol{P}_\pi\times\boldsymbol{R}), \quad (27c)$$

$$\mu_v=-M\int[\phi\times(z\times\Gamma\phi)]_3 dz+\frac{1}{2}(\boldsymbol{R}\times\int(\phi\times\Gamma\phi)_3 dz+\text{h.c.})$$

$$+\tau_3\left(\frac{\sigma}{2}\right)+\left(\frac{H_\pi}{M}\right)\frac{\sigma}{2}\tau_3+\left(\frac{1}{2M}\right)\frac{\tau_3}{2}(\boldsymbol{R}\times\boldsymbol{P}_\pi+\text{h.c.})$$

$$+\left(\frac{f_0}{2M}\right)(\boldsymbol{\tau}\times\boldsymbol{\phi})_3(\boldsymbol{\sigma}\times\boldsymbol{R}). \quad (27d)$$

The first terms in (27) and the third term of (27d) are the ones given in the static model and the others are the relativistic corrections. The second term in (27d) combined with the first term has a clear-cut classical image. If we notice that $\boldsymbol{j}_\pi(z)=-e_0\phi\times\Gamma\phi$ is defined in the coordinate system with the origin at the core nucleon, we can write the magnetic moment μ_π due to the meson cloud as

$$\mu_\pi=\tfrac{1}{2}\int z\times\boldsymbol{j}_\pi(z-\boldsymbol{r}_c)\,dz,$$

where $\boldsymbol{r}_c$ is the coordinate of the core nucleon in the center of mass system and is given by, according to Suura,[11]

$$\boldsymbol{r}_c=-\boldsymbol{R}/M.$$

Then, we have

$$\mu_\pi=\left(\frac{e_0}{2M}\right)\left\{M\int \boldsymbol{z}\times \boldsymbol{j}_\pi(\boldsymbol{z})\,d\boldsymbol{z}-\frac{1}{M}\boldsymbol{R}\times\int \boldsymbol{j}_\pi(\boldsymbol{z})\,d\boldsymbol{z}\right\},$$

which is exactly the sum of the first and second terms of (27d). The last term of (27c) and the fifth term of (27d) are the corrections corresponding to the core recoil discussed by Holladay[3] in a pure kinematical manner. The second term in (27b) can be written in a form

$$[\boldsymbol{r}_c,\ \int \boldsymbol{j}_\pi(\boldsymbol{z})\,d\boldsymbol{z}],$$

except a certain constant factor, then it can be interpreted as the charge induced by the distortion of the current of the field by the nucleon recoil. The other terms are originated with the zitterbewegung of the core nucleon and it is difficult to sketch simple physical pictures.

The requirement of the gauge invariance enforces that the real neutron has no charge, then the neutron expectation value of the total charge operator $Q=Q_s+Q_v$ must vanish.

$$\langle n|Q|n\rangle=\langle n|Q_s|n\rangle+\langle n|Q_v|n\rangle=0. \tag{28}$$

In the covariant γ_5-theory this condition (28) is satisfied exactly in each powers of coupling constant f_0. In our case, however, it is expected to satisfy the condition only approximately. It is because the higher order effects are included partially in our calculation. Moreover the neglect of S-wave pair term and the introduction of the cut-off procedure may violate the condition (28). If we define the quantity

$$J\equiv\frac{\langle n|Q|n\rangle}{\langle p|Q|p\rangle}, \tag{29}$$

this gives a measure for the violation of the gauge invariance. It is shown in Table I that J is satisfactorily small for the reasonable choice of the coupling constant and cut-off momentum. In the static model the condition (28) is not satisfied at all.

If the condition (28) holds, the renormalized proton charge is given by

$$e=e_0\langle p|Q|p\rangle=2e_0\langle p|Q_s|p\rangle. \tag{30}$$

As is seen in (27a) Q_s is exactly one half up to the order of $1/M$, then the renormalized charge is equal to the unrenormalized charge in our approximation.

To calculate the expectation values of each term of (27), we neglect the $(1/M_n)^2$ term in (22). This neglect gives a certain effect of order $1/M$ to the expectation values of μ_n, but it may not be important numerically. As for the recoil energy $\boldsymbol{P}_\pi^2/2M$ we keep only the diagonal parts in meson occupation numbers specified by angular momentum. If we expand the meson fields as usual:

$$\phi_\alpha(\boldsymbol{x})=\sum_{k,l,m}(2\omega_k)^{-1/2}[a^\alpha_{klm}+a^{\alpha *}_{kl-m}]\psi_{k,l,m}(\boldsymbol{x}),$$

$$\boldsymbol{\pi}_\alpha(\boldsymbol{x})=\sum_{k,l,m}(-i)\left(\frac{\omega_k}{2}\right)^{1/2}[a^\alpha_{klm}-a^{\alpha *}_{kl-m}]\psi_{k,l,m}(\boldsymbol{x}), \tag{31}$$

the final Hamiltonian we use is

$$H=\sum_{k,l,m}\omega_k{}^* a^{\alpha *}_{klm}a^\alpha_{klm}+\left(\frac{f_0}{2M}\right)\frac{1}{\sqrt{6\pi R}}\sum_k\frac{k^2U(k)}{\sqrt{2\omega_k}}\sigma_m\tau_\alpha(a^\alpha_{klm}+a^{\alpha *}_{kl-m}), \tag{32}$$

where

$$\psi_{klm}(\boldsymbol{x})=(2/R)^{1/2}kj_l(kr)Y_{lm}(\theta,\ \varphi), \tag{33}$$

and

$$\omega_k{}^*=\omega_k+\frac{k^2}{2M}. \tag{34}$$

R is the radius of the normalization sphere and $U(k)$ is the cut-off function in momentum space.

We can reduce the expectation value of each term in (27), which contains the meson quantities, to the expectation values of bilinear products of the P-wave meson operators, since the real nucleon state $|\rangle$ contains only P-wave mesons in the cloud. The result of such a reduction is:

$$-M\int[\phi\times(\boldsymbol{z}\times\Gamma\phi)]_z d\boldsymbol{z}=M\sum_{k,m}\frac{m}{\omega_k}(a^{+*}_{km}a^-_{km}-a^{-*}_{km}a^-_{km}+a^{-*}_{km}a^{-*}_{k-m}+a^+_{km}a^-_{k-m}), \tag{35a}$$

$$\frac{1}{2}(\boldsymbol{R}\times\int(\phi\times\Gamma\phi)_z d\boldsymbol{z}+\text{h.c.})=-\sum_{k,m}m(a^{+*}_{km}a^+_{km}-a^{-*}_{km}a^-_{km}+a^{+*}_{km}a^{-*}_{k-m}+a^+_{km}a^-_{k-m}), \tag{35b}$$

$$\frac{2}{3}\int(\boldsymbol{\pi}\times z\Gamma\phi)_x d\boldsymbol{z}=-\frac{i}{3}\begin{pmatrix}\pi\\ R\end{pmatrix}\sum_{kk'm}U(k)U(k')\left(\frac{\omega_k}{\omega_{k'}}\right)^{1/2}[(\boldsymbol{a}_{km}-\boldsymbol{a}^*_{k-m})$$

$$\times(\boldsymbol{a}_{k'-m}+\boldsymbol{a}^*_{k'm})]_z\times\left(k'\frac{d}{dk'}\delta(k-k')-\delta(k-k')\right), \tag{35c}$$

$$\frac{1}{3M}\boldsymbol{P}_\pi\int(\phi\times\Gamma\phi)_z d\boldsymbol{z}=i\left(\frac{\pi}{R}\right)^2\sum_{kk'k''m}\left(\frac{k^2}{\omega_k}\right)[\boldsymbol{a}^*_{km}\times\boldsymbol{a}_{km}]_z\delta(k-k')\delta(k-k''), \tag{35d}$$

$$\left(\frac{H_\pi}{M}\right)\sigma\left(\frac{1+\tau_3}{2}\right)=\sigma\left(\frac{1+\tau_3}{2}\right)\frac{1}{M}\sum_{k,m}\omega_k\boldsymbol{a}^*_{km}\boldsymbol{a}_{km}, \tag{35e}$$

$$\left(\frac{1}{2M}\right)\left(\frac{1+\tau_3}{2}\right)(\boldsymbol{R}\times\boldsymbol{P}_\pi-\boldsymbol{P}_\pi\times\boldsymbol{R})=\left(\frac{1+\tau_3}{2}\right)\frac{1}{M}\sum_{k,m}m\omega_k\boldsymbol{a}^*_{km}\boldsymbol{a}_{km}, \tag{35f}$$

where $\delta(k-k')$ means $(R/\pi)\delta_{kk'}$. Only exception is the last term of (27d), which is reduced to a form linear in the meson operator.

The expectation values of the products can be obtained from Miyazawa's result[1] with slight modification coming from the replacement of ω_k by $\omega_k{}^*$ in our Hamiltonian. These values consist of the perturbation term with the renormalized coupling constant and the integral terms containing higher order effects through the scattering cross section. In

evaluating the relativistic correction terms we neglect the latter integral terms, which are always 20～30% of the perturbation term. The expectation value of the last term of (27d) can also be divided into two terms, the perturbation term and the integral term and we neglect the latter term in this case, too. With this approximation, the expectation values of each term of (27) are given by

$$Q_{v2} \equiv \langle | \left(\frac{2}{3}\right) \int (\boldsymbol{\pi} \times z \Gamma \phi)_3 \, dz | \rangle = \left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \left(\frac{4}{3\pi}\right) \left[4K_{1,2}^4 - K_{3,2}^6 - \frac{1}{2} K_{2,3}^6 - \frac{1}{2M} K_{1,3}^6 + L_{1,2}^5 \right],$$

$$Q_{v3} \equiv \langle | \left(\frac{1}{3M}\right) \boldsymbol{P}_\pi \cdot \int (\phi \times \Gamma \phi)_3 \, dz | \rangle = -\left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \left(\frac{2}{3\pi}\right) \frac{1}{M} K_{1,2}^4, \tag{36}$$

$$\mu_{s2} \equiv \langle | \left(\frac{1}{2M}\right) H_\pi \sigma_z | \rangle = \left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \langle \sigma_z \rangle \left(\frac{3}{2\pi}\right) \frac{1}{M} K_{0,2}^4,$$

$$\mu_{s3} \equiv \langle | \left(\frac{1}{4M}\right) (\boldsymbol{R} \times \boldsymbol{P}_\pi - \boldsymbol{P}_\pi \times \boldsymbol{R}) | \rangle = \left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \frac{1}{\pi} \frac{1}{M} K_{0,2}^4,$$

$$\mu_{v2} \equiv \langle | \left(\frac{1}{2}\right) (\boldsymbol{R} \times \int (\phi \times \Gamma \phi)_3 dz + \text{h.c.}) | \rangle = -\left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \frac{8}{3\pi} K_{1,2}^4,$$

$$\mu_{v4} \equiv \langle | \left(\frac{1}{2M}\right) H_\pi \tau_3 \sigma_z | \rangle = \left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \langle | \tau_3 \sigma_z | \rangle \left(\frac{3}{2\pi}\right) \frac{1}{M} K_{0,2}^4,$$

$$\mu_{v5} \equiv \langle | \left(-\frac{1}{4M}\right) \tau_3 (\boldsymbol{R} \times \boldsymbol{P}_\pi - \boldsymbol{P}_\pi \times \boldsymbol{R})_z | \rangle = \left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \langle \tau_3 \rangle \frac{1}{\pi M} K_{1,2}^4,$$

$$\mu_{v6} \equiv \langle | \left(\frac{f_0}{2M}\right) (\tau \times \phi)_3 (\sigma \times \boldsymbol{R})_z | \rangle = -\left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \left(\frac{2}{3\pi}\right) L_{0,1}^3,$$

$$\mu_{v1} \equiv \langle | (-M) \int [\phi \times (z \times \Gamma \phi)]_{3,z} dz | \rangle = \left(\frac{g^2}{4\pi}\right) \frac{1}{\mu^2} \left(\frac{8}{3\pi}\right) M K_{2,2}^4 + \left(\frac{4M}{9}\right) (I_2^* + I_3^*),$$

for the proton spin-up state, where

$$K_{\alpha,\beta}^{\alpha} = \int \frac{k^\alpha dk}{\omega^\beta \omega^{*\gamma}} \quad \text{and} \quad L_{\alpha,\beta}^{\alpha} = \int \frac{k^\alpha U^\gamma(k)}{\omega^\beta \omega^{*\gamma}} \frac{dU^\gamma(k)}{dk} dk.$$

I_2^* and I_3^* are the double integrals, containing the cross section, i. e.,

$$I_2^* = \frac{1}{12\pi^3} \int \frac{(\sigma_{33}(k) - \sigma_{31}(k) - \sigma_{13}(k) + \sigma_{11}(k)) v_k^* \omega_k l^4 dk dl}{\omega_l^2 (\omega_k^* + \omega_l^*)^2 k^2},$$

$$I_3^* = \frac{1}{12\pi^3} \int \frac{(\sigma_{33}(k) - \sigma_{31}(k) - \sigma_{13}(k) + \sigma_{11}(k)) v_k^* \omega_k l^4 dk dl}{\omega_l^2 \omega_l^* (\omega_k^* + \omega_l^*) k^2}.$$

v_k^* is the relative velocity in the center of mass system. g is the equivalent $ps-pv$ coupling constant. The expectation value of τ_3 or σ_3 is given in two ways,[1] corresponding to eq. (23) or (28) in Miyazawa's paper:

$$\text{A}: \quad Q_{v1A} \text{ or } \mu_{s1A} \equiv \langle | \frac{\tau_3}{2} | \rangle = \langle | \frac{\sigma_3}{2} | \rangle = \frac{1}{2} - \left(\frac{g^2}{4\pi} \right) \frac{2}{\pi} K_{1,2}^4 + I_1^*, \tag{37a}$$

$$\text{B}: \quad Q_{v1B} \text{ or } \mu_{s1B} \equiv \langle | \frac{\tau_3}{2} | \rangle = \langle | \frac{\sigma_3}{2} | \rangle = \frac{1}{2} - \frac{1}{g_0^2} \frac{1}{2\pi} \int \frac{\sigma^+ \omega_k v_k^{*2} dk}{k^2}, \tag{37b}$$

where the intergal I_1^* is given by

$$I_1^* = \frac{1}{12\pi^3} \int \frac{(\sigma_{33}(k) - \frac{1}{2}\sigma_{31}(k) - \frac{1}{2}\sigma_{11}(k)) v_k^* \omega_k l^4 dk dl}{\omega_l (\omega_k^* + \omega_l^*)^2 k^2}$$

and

$$g_0^2 = g^2 + \frac{\mu^2}{2\pi} \int \frac{\sigma^+ + \sigma^-}{k^2} \omega_k v_k^{*2} dk \tag{38}$$

is the unrenormalized coupling constant. $\langle | \sigma_3 \tau_3 | \rangle$ is also given by Miyazawa, i. e.,

$$\frac{1}{2} \langle | \sigma_3 \tau_3 | \rangle = \frac{1}{2} \left(\frac{g}{g_0} \right) \equiv \mu_{v3}. \tag{39}$$

§ 4. Numerical results and discussions

We calculate the quantities in (36), (37) and (38) with the straight cut of the meson momentum, which may produce the ambiguity of a few per cent in the numerical values. The results are given in the following tables for the several choice of the coupling constant and the cut-off momentum.

Table Ia. The nucleon charge and the values of Δ for the case of $g^2/4\pi = 0.08$.

$k_{\max}$	Q_{v1}		Q_{v2}	Q_{v3}	Total Q		Δ	
	A	B			A	B	A	B
5μ	0.303	0.097	0.359	−0.055	0.607	0.401	−0.097	+0.110
6μ	0.269	0.087	0.507	−0.091	0.685	0.503	−0.156	−0.003
7μ	0.253	0.080	0.668	−0.128	0.783	0.611	−0.221	−0.100

Table Ib. The same quantities as the above for the case of $g^2/4\pi = 0.10$.

$k_{\max}$	Q_{v1}		Q_{v2}	Q_{v3}	Total Q		Δ	
	A	B			A	B	A	B
5μ	0.219	0.124	0.449	−0.069	0.599	0.504	−0.009	−0.004
6μ	0.154	0.113	0.634	−0.114	0.673	0.613	−0.147	−0.102
7μ	0.099	0.105	0.836	−0.173	0.762	0.768	−0.208	−0.211

i) *Nucleon current (charge)*

The vector parts of the nuclear charge are given in Tables I.

A or B refers to (36a) or (36b), corresponding to the two different ways to calculate $\langle|\tau_3|\rangle$ or $\langle|\sigma_3|\rangle$. Q_{v1} is overestimated slightly for the case A, since we have neglected all σ_{ij} except σ_{zz} in evaluating $I^{\alpha\beta}$'s. This neglect makes $I^{\alpha\beta}$ definitely large. These values of Δ seem to be satisfactory in the qualitative sense.

ii) *Scalar part of the magnetic moment*

The scalar parts of the magnetic moment are given in Tables II.

Table IIa. The scalar parts of the magnetic moment for the case of $g^2/4\pi=0.08$.

k_{max}	μ_{s1}		μ_{s2}		μ_{s3}	Total μ_s	
	A	B	A	B		A	B
5μ	0.303	0.097	0.082	0.026	0.090	0.475	0.213
6μ	0.269	0.087	0.118	0.038	0.147	0.534	0.272
7μ	0.253	0.080	0.166	0.052	0.219	0.538	0.351

Table IIb. The same quantities as the above for the case of $g^2/4\pi=0.10$.

k_{max}	μ_{s1}		μ_{s2}		μ_{s3}	Total μ_s	
	A	B	A	B		A	B
5μ	0.219	0.124	0.074	0.042	0.113	0.406	0.279
6μ	0.154	0.113	0.085	0.062	0.183	0.422	0.358
7μ	0.099	0.105	0.082	0.086	0.273	0.454	0.464

The empirical value is $\mu_s=0.439$. The theoretical values are also in agreement with the empirical one qualitatively.

iii) *Vector parts of the magnetic moment*

The vector parts of the moment are given in Tables III.

The empirical value is $\mu_v=2.35$. The theoretical values are certainly less than the empirical one, though it is not a discouraging discrepancy. The disagreement is due to two facts. First the modification $\omega_k \to \omega_k^*$ reduces the value of μ_v from Miyazawa's value (by about 30% in the case of $f^2/4\pi=0.08$ and $k_{max}=6\mu$). Second the main correction μ_{v2} is destructive and not so small. As is mentioned in the previous section

Table IIIa. The vector part of the magnetic moment for the case of $g^2/4\pi=0.08$.

k_{max}	μ_{v1}	μ_{v2}	μ_{v3}	μ_{v4}	μ_{v5}		μ_{v6}	Total μ_v	
					A	B		A	B
5μ	1.145	−0.447	0.272	0.073	0.055	0.018	0.156	1.25	1.22
6μ	1.462	−0.627	0.262	0.115	0.079	0.025	0.209	1.50	1.45
7μ	1.770	−0.821	0.253	0.166	0.110	0.035	0.271	1.75	1.67

Table IIIb. The same quantities as the above for the case of $g^2/4\pi=0.10$.

$k_{\max}$	μ_{v1}	μ_{v2}	μ_{v3}	μ_{v4}	μ_{v5}		μ_{v6}	Total μ_v	
					A	B		A	B
5μ	1.376	−0.559	0.294	0.099	0.050	0.028	0.194	1.45	1.43
6μ	1.749	−0.784	0.283	0.156	0.056	0.041	0.261	1.72	1.71
7μ	2.105	−1.026	0.274	0.225	0.054	0.057	0.339	1.97	1.97

there is a bit of ambiguity of order $1/M$ in calculating μ_v, but it may not alter this situation essentially.

In conclusion we summarize the results as follows: first, the relativistic corrections are very important for the nucleon current and the magnetic moment, especially for the vector part of the current and the scalar part of the moment. The results are roughly in agreement with experiment, except for the vector part of the moment, with suitable choice of the coupling constant ($g^2/4\pi=0.08\sim0.10$) and the cut-off momentum ($k_{\max}=5\mu\sim7\mu$). The vector part is about a few ten per cent smaller than the empirical value, and this comes from the reduction of the main term and the destructive effect of the main correction. Second, the renormalization of the electric charge and the nucleon mass are both considered in this paper. The renormalization of the nucleon mass has no effects within the approximation made here. As to the charge renormalization it was shown that the renormalized charge is equal to the unrenormalized charge up to the order of $1/M$, if the neutron charge vanishes. The vanishing of the neutron charge was realized only approximately.

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Appendix

Here we shall derive the relations (25). To justify (25a), it is enough to see

$$\langle|A_{ij}|\rangle\equiv\langle|\int(\pi\times\nabla_i\phi)_3 z_j dz|\rangle=\alpha\delta_{ij}, \tag{A·1}$$

where α is a pure numerical constant. We consider the transformation of the space coordinate from (i, j, k) to (j, i, k), and refer the operator corresponding to this transformation as S. Then, the manipulation below can easily follow:

$$\begin{aligned}\langle|A_{ij}|\rangle&=\langle|\int(\pi\times\nabla_i\phi)_3 z_j dz|\rangle=\langle|S^{-1}S\int(\pi\times\nabla_i\phi)_3 z_j dz S^{-1}S|\rangle\\&=\langle|\int(\pi'(z')\times\nabla_i'\phi'(z'))_3 z_j' dz'|\rangle=\langle|\int(\pi(z)\times\nabla_j\phi(z))_3 z_i dz|\rangle\\&=\langle|A_{ji}|\rangle.\end{aligned} \tag{A·2}$$

Since the only Hermitian symmetric tensor in the charge space is $\tau_i\tau_j+\tau_j\tau_i$ or δ_{ij}, this ensures the equation (A·1).

To justify (25b), it is enough to show the relation,

$$\langle|B_{ij}|\rangle \equiv \langle|\{R_i, \int(\phi\times\Gamma_j\phi)_3 dz\}|\rangle = \beta\langle u|(\sigma_i\sigma_j-\sigma_j\sigma_i)|u\rangle, \tag{A·3}$$

where $|u\rangle$ is the bare spinor. In general, it can be written as

$$\langle|R_i\int(\phi\times\Gamma_j\phi)_3 dz|\rangle = \langle u|(\beta\sigma_i\sigma_j+\gamma\delta_{ij})|u\rangle, \tag{A·4}$$

where β and γ are the pure constants. The simple calculation shows

$$\begin{aligned}\langle|R_i\int(\phi\times\Gamma_j\phi)_3 dz|\rangle^+ &= \langle u|(\beta^*\sigma_j\sigma_i+\gamma^*\delta_{ij})|u\rangle \\ &= \langle u|(R_i\int(\phi\times\Gamma_j\phi)_3 dz+2i\int(\pi\times\Gamma_i\phi)_3 z_j dz)|u\rangle \\ &= \langle u|(\beta\sigma_i\sigma_j+\gamma\delta_{ij}+2i\alpha\delta_{ij})|u\rangle.\end{aligned} \tag{A·5}$$

(A·5) implies

$$\beta = -\beta^* \tag{A·6}$$

Next, we apply Wigner time reversal T_w to $\langle|B_{ii}|\rangle$, i. e.,

$$\begin{aligned}\langle s|B_{ii}|s'\rangle &= \langle u(s)|(\gamma+\gamma^*)|u(s')\rangle \\ &= \langle s|T_w^{-1}T_wB_{ii}T_w^{-1}T_w|s'\rangle = -\langle -s'|B_{ii}|-s\rangle \\ &= -\langle u(-s')(\gamma-\gamma^*)u(-s)\rangle.\end{aligned} \tag{A·7}$$

Putting $s'=s$ in (A·7), we have

$$\gamma = -\gamma^* \tag{A·8}$$

(A·6) and (A·8) ensure the relation (A·3). The other relations can also be verified in the similar manner.

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On the Exchange Magnetic Moments

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The exchange magnetic moment operators for the two nucleon system are obtained in terms of photopion production matrix elements. This photopion production part is closely related with the pion-nucleon scattering part, thus enabling us to use $\pi-N$ scattering experimental values in the expressions of the exchange magnetic moment operators. These formulae are applied to cases of triton magnetic moment and those of heavy nuclei (the Fermi gas model). In the former case, rather a good result is obtained, while in the latter it is found that exchange magnetic moments can explain only small parts of the deviations from the Schmidt lines.

§ 1. Introduction

Many attempts have so far been made both phenomenologically[1] and theoretically[2),3)] in an effort to derive possible forms of exchange magnetic moment operators which come from virtual pion currents between nucleons. Every theoretical approach employed for this purpose, however, has been based on the perturbation expansion method in the coupling constant.

A different approach is here made to this problem. It involves an expansion only in the number of pions exchanged, and the pion self field of each nucleon is treated, in principle, exactly. In this paper we compute the exchange magnetic moment involving the interchange of one pion.

The process of our interest as a whole can be regarded as photopion production at one of the nucleons followed by the exchange of that pion with another. The matrix element of the photopion production process has been studied by Chew and Low[4] based on the static pion theory. Their analysis indicates that the matrix element is composed of two parts, one arising from pion current and the other from nucleon current.* The exchange moment coming from the former part, the pion current term, is essentially equal to the result of the lowest order perturbation theory, which has already been calculated.

In § 2 we obtain the contribution from the nucleon current using our method, and the correction to the triton magnetic moment is evaluated in § 3. § 4 is devoted to the calculation of the exchange moment in the case of heavy nuclei by the Fermi gas model.

§ 2. Exchange magnetic moment due to nucleon current

In this section we shall derive the magnetic moment operator arising from the

* This division of current is different from the conventional one. See ref. 4).

nucleon current by means of the S-matrix calculation. Before that, however, it is necessary to summarize the result of pion current contribution to the exchange magnetic moment. Chew and Low have shown that the photopion matrix element that comes from the pion current is essentially equal to the usual renormalized Born approximation if small terms are neglected. Under this approximation the exchange magnetic moment from this part agrees completely with that of previous calculations using the perturbation method in the lowest order.

Using the static approximation throughout and restricting the number of pions exchanged to one, the exchange magnetic moment operators due to pion currents for the two nucleon system can be derived in the following way. The processes we consider here are the interaction of the photon with the charged pion currents existing between a pair of nucleons and the interaction with the so-called interaction currents. This term, which is evidently connected with the former on the gauge invariance requirement, is written as :*

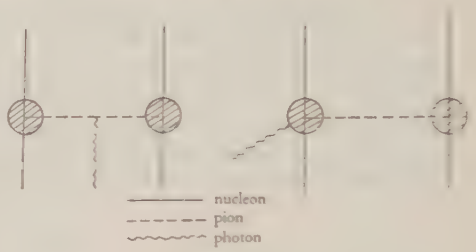

Fig. 1

$$-\boldsymbol{j}_{\text{int}}\cdot\boldsymbol{A}=-ef(\tau_1\phi_2-\tau_2\phi_1)\,(\sigma\boldsymbol{A}).$$

The diagrams for these processes are shown in Fig. 1. Note that in the latter process all of the photon, pion and nucleon interact precisely at one point of space. S-matrix for the first of these diagrams is this,

$$\begin{aligned}S=&-i\frac{e\boldsymbol{H}}{2}\frac{f^2}{(2\pi)^2}[\boldsymbol{\tau}^{(1)}\times\boldsymbol{\tau}^{(2)}]_3\delta(0)\\&\times\int d^3k\,d^3q\Big[(\sigma^{(1)}\boldsymbol{k})\,(\sigma^{(2)}\boldsymbol{q})\,\{(k_1+q_1)\Gamma_2\delta^3(k-q)\\&-(k_2+q_2)\Gamma_1\delta^3(k-q)\}\frac{1}{\omega_k^2\omega_q^2}e^{i(\boldsymbol{q}\boldsymbol{x}^{(2)}-\boldsymbol{k}\boldsymbol{x}^{(1)})}\Big]\end{aligned}$$

which yields after a little calculation

$$\begin{aligned}=&-i\frac{e\boldsymbol{H}}{4}f^2[\boldsymbol{\tau}^{(1)}\times\boldsymbol{\tau}^{(2)}]_3\delta(0)\,\{\sigma_1^{(2)}x_2(\sigma^{(2)}\Gamma)\,Y-\sigma_2^{(1)}x_1(\sigma^{(2)}\Gamma)\,Y\\&+[\sigma^{(1)}\times\sigma^{(2)}]_3\,Y-[\boldsymbol{x}^{(1)}\times\sigma^{(1)}]_3(\sigma^{(2)}\Gamma)\,Y\\&-[\boldsymbol{x}^{(1)}\times\sigma^{(2)}]_3(\sigma^{(1)}\Gamma)\,Y-[\boldsymbol{x}^{(1)}\times\boldsymbol{x}]_3V(12)\}.\end{aligned}$$

In the same manner, we get the contribution from the second diagram.

$$\begin{aligned}S=&-i\frac{e\boldsymbol{H}}{4}f^2[\boldsymbol{\tau}^{(1)}\times\boldsymbol{\tau}^{(2)}]_3\delta(0)\\&\times\{[\boldsymbol{x}^{(2)}\times\sigma^{(2)}]_3(\sigma^{(1)}\Gamma)\,Y-[\boldsymbol{x}^{(1)}\times\sigma^{(1)}]_3(\sigma^{(2)}\Gamma)\,Y\}.\end{aligned}$$

* We take $\hbar=c=\mu=1$; μ is the pion rest mass.

In the above expressions $\boldsymbol{x}^{(i)}$ is the position vector of the i-th nucleon and $\boldsymbol{x}$ denotes the inter-nucleon distance with the magnitude x. Also f is the renormalized coupling constant and Y means the Yukawa potential. All the other notations will be understood without explanation.

Combining these results and noting that the quantity, which multiplies the factor $2\pi i\delta(0)\boldsymbol{H}$ where $\boldsymbol{H}$ is the magnetic field, makes the magnetic moment operator, we get after some more calculations the following expressions for the exchange magnetic moment arising from the pion current.

$$\boldsymbol{M}^{(\pi)}_{\text{exch.}} = -\frac{ef^2}{8\pi}\left[\boldsymbol{\tau}^{(1)}\times\boldsymbol{\tau}^{(2)}\right]_3 \times\left[\left\{\frac{x_3(\boldsymbol{x}\cdot\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)})}{x^2}\left(1+\frac{1}{x}\right)-\left[\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)}\right]_3\right\}e^{-x} +\frac{1}{2}\left\{(\boldsymbol{x}^{(1)}+\boldsymbol{x}^{(2)})\times\boldsymbol{x}\right\}_3 V(12)\right] \tag{1}$$

where

$$V(12)=\left[\frac{1}{3}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})+\left(\frac{1}{3}+\frac{1}{x}+\frac{1}{x^2}\right)T_{12}\right]\frac{e^{-x}}{x}$$

$$T_{12}=3\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{x})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{x})}{x^2}-(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}).$$

No consideration has been made in the above evaluation of the pion-nucleon interaction expected to occur at the different places along the nucleon line.

Next, and this being the main subject in this section, we consider the effect resulting from the nucleon current. The photon interaction with this current is composed of both scalar and vector parts with respect to the isotopic spin space, and it has been shown by Chew and Low that the former is negligibly smaller than the latter. Therefore, we do not consider the scalar part here. The vector part of the interaction is of the form $i(\mu_p-\mu_n)/2\cdot\tau_3(\boldsymbol{\sigma H})$, and the comparison with the pseudovector coupling term $f\tau_\alpha(\boldsymbol{\sigma k})$ of the pion-nucleon system indicates the possibility of replacing photon by a neutral pion with charge index 3, momentum $\boldsymbol{H}$ and coupling constant $i(\mu_p-\mu_n)/2$.

With this interpretation, we can treat the photopion production part in the same way as the pion-nucleon scattering process. The scattering matrix for the P-wave pion from state $(i\boldsymbol{k})$ to state $(j\boldsymbol{q})$ by a fixed nucleon is given by[5]

$$\langle j\boldsymbol{q}|s|i\boldsymbol{k}\rangle=2\pi i\delta(k_0-q_0)\Big[A(k_0)\tau_i\tau_j(\boldsymbol{\sigma k})(\boldsymbol{\sigma q}) +B(k_0)\{\tau_i\tau_j(\boldsymbol{\sigma q})(\boldsymbol{\sigma k})+\tau_j\tau_i(\boldsymbol{\sigma k})(\boldsymbol{\sigma q})\} +C(k_0)\tau_j\tau_i(\boldsymbol{\sigma q})(\boldsymbol{\sigma k})\Big]e^{i(k-q)x}.$$

$(i\boldsymbol{k})$ are the charge index and momentum of the incident pion and $(j\boldsymbol{q})$ are those of

the scattered pion. Values of A, B and C for zero-energy ($k_0=0$), which are necessary here, are given by

$$A(0)=C(0)=\frac{5}{18\pi}\int_0^\infty \frac{dp}{\omega_p^2}\sigma_{33}(p)$$

$$B(0)=\frac{1}{6\pi}\int_0^\infty \frac{dp}{\omega_p^2}\sigma_{33}(p)$$

where σ_{33} is the total cross section of the pure 3/2, 3/2 state. In this expression, σ_{11} and σ_{13} are neglected. The numerical value of the above integral is

$$\int_0^\infty \frac{\sigma_{33}}{\omega_p^2}\,dp=3.7.$$

The most striking feature of this scattering matrix element is that it can be computed using the experimental value of scattering cross section. By virtue of this fact, every effect occurring at the scattering part can be included in our result.

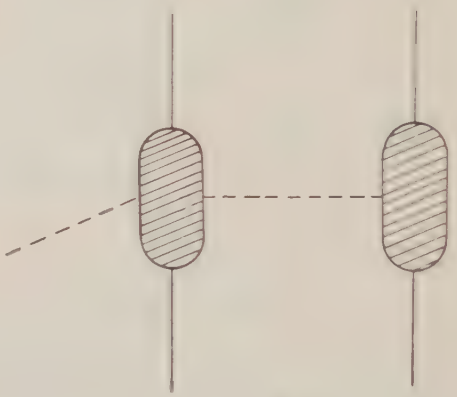

Fig. 2

The S-matrix for our process is given by

$$S=i\frac{1}{f}\frac{\mu_p-\mu_n}{2}\int d^4q\langle j\boldsymbol{q}|s|3,\boldsymbol{H}\rangle$$

$$\times 2\pi\delta(q_0)f\tau_j^{(2)}(\sigma^{(2)}\boldsymbol{q})\,e^{i\boldsymbol{q}\boldsymbol{x}^{(2)}}\cdot\frac{i}{(2\pi)^4}\frac{1}{q_0^2-\omega_q^2}$$

+terms with nucleon numbers interchanged,

which describes the processes that the external photon (considered as the equivalent pion), interacting with one of the nucleons, produces the virtual pion, which is subsequently absorbed by another nucleon. The Feynman diagram of this process is shown in Fig. 2. After a little calculation of this S-matrix, we obtain the final result of the exchange magnetic moment due to the nucleon current.

They are:

$$\boldsymbol{M}_{\text{exch.}}^{(N)}=\sum_{i=1}^{6}\boldsymbol{M}_i \tag{2}$$

where

$$\boldsymbol{M}_1=(\tau_3^{(1)}+\tau_3^{(2)})\left\{3\frac{\boldsymbol{x}(\boldsymbol{x}\cdot\sigma^{(1)}+\sigma^{(2)})}{x^2}-(\sigma^{(1)}+\sigma^{(2)})\right\}f_1(x)$$

$$\boldsymbol{M}_2=(\tau_3^{(1)}+\tau_3^{(2)})\,(\sigma^{(1)}+\sigma^{(2)})\,f_2(x)$$

$$\boldsymbol{M}_3=(\tau_3^{(1)}-\tau_3^{(2)})\left\{3\frac{\boldsymbol{x}(\boldsymbol{x}\cdot\sigma^{(1)}-\sigma^{(2)})}{x^2}-(\sigma^{(1)}-\sigma^{(2)})\right\}f_3(x)$$

$$\boldsymbol{M}_4=(\tau_3^{(1)}-\tau_3^{(2)})\,(\sigma^{(1)}-\sigma^{(2)})\,f_4(x)$$

$$\boldsymbol{M}_5=[\boldsymbol{\tau}^{(1)}\times\boldsymbol{\tau}^{(2)}]_3\left\{3\frac{\boldsymbol{x}(\boldsymbol{x}\cdot\sigma^{(1)}\times\sigma^{(2)})}{x^2}-[\sigma^{(1)}\times\sigma^{(2)}]\right\}f_5(x)$$

$$\boldsymbol{M}_6=[\boldsymbol{\tau}^{(1)}\times\boldsymbol{\tau}^{(2)}]_3[\sigma^{(1)}\times\sigma^{(2)}]f_6(x)$$

and

$$f_1(x)=f_3(x)=-\frac{2}{15\pi}\,\frac{\mu_p-\mu_n}{2}\,A(0)\left(\frac{3}{x^3}+\frac{3}{x^2}+\frac{1}{x}\right)e^{-x}$$

$$f_2(x)=f_4(x)=f_6(x)=-\frac{2}{15\pi}\,\frac{\mu_p-\mu_n}{2}\,A(0)\,\frac{e^{-x}}{x}$$

$$f_5(x)=-\frac{1}{15\pi}\,\frac{\mu_p-\mu_n}{2}\,A(0)\left(\frac{3}{x^3}+\frac{3}{x^2}+\frac{1}{x}\right)e^{-x}.$$

Thus, the total exchange moment is given by the sum

$$\boldsymbol{M}=\boldsymbol{M}_{\text{exch.}}^{(\pi)}+\boldsymbol{M}_{\text{exch.}}^{(N)}.$$

§ 3. Application to the triton magnetic moment

As the first application of the results obtained in the last section, we consider the triton magnetic moment. As is well known, its experimental value is larger than that in the Schmidt model, and the admixture of states in the triton ground state cannot be responsible for this deviation since it has the wrong sign for this effect. So we want to see to what extent the pion effect could explain this disagreement. Assuming that only one pion is exchanged between a pair of nucleons and that none between the other pairs at the same time, the magnetic moment operators for the three nucleon system can be given by

$$\boldsymbol{M}(123)=\boldsymbol{M}(12)+\boldsymbol{M}(23)+\boldsymbol{M}(13).$$

Now the expectation value of $\boldsymbol{M}^{(\pi)}$ for the three nucleon case with respect to its ground state gives the numerical value $\simeq+0.3$ n. m. and this makes a part of corrections due to the exchange pion current.[2)]

To obtain a further correction, we must evaluate the expectation values of the operators listed in eq. (2) (nucleon current contribution). For the triton wave function we take

$$\Phi=\frac{1}{\sqrt{6}}\begin{vmatrix}\alpha(1)n(1) & \alpha(2)n(2) & \alpha(3)n(3)\\ \alpha(1)p(1) & \alpha(2)p(2) & \alpha(3)p(3)\\ \beta(1)n(1) & \beta(2)n(2) & \beta(3)n(3)\end{vmatrix}\varphi(123)$$

where α, β and p, n are spin and isotopic spin functions respectively. $\varphi(123)$ is the part of space variables, totally symmetric with respect to inter-nucleon distances x_{12}, x_{23} and x_{31}, and assumed to be of the following type :[6]

$$\varphi(123)=\exp[-\lambda(x_{12}+x_{23}+x_{31})]$$

where λ is a parameter with the numerical value of 0.5033.

Calculations are easy and we obtain the result that the contribution of M_{ex}^{N} terms to the triton magnetic moment is about -0.062 n.m. This value, when added to the above mentioned one which is the same as Villar's result, forms evidently the total modification to the triton magnetic moment caused by the exchange currents.

§ 4. Exchange magnetic moments of heavy nuclei

In this section, we calculate exchange magnetic moments of heavy nuclei. For the nuclear wave function, we choose the Fermi-gas model wave function Ψ, since it is the most convenient one for calculations and since our main purpose of this calculation is to see the degree of the deviation of nuclear magnetic moments from the Schmidt lines. The function is

$$\Psi=\begin{vmatrix}\varphi_1(1), & \cdots\cdots, & \varphi_N(1)\\ & \cdots\cdots & \\ & \cdots\cdots & \\ \varphi_1(N), & \cdots\cdots, & \varphi_N(N)\end{vmatrix}$$

where

$$\varphi_i(1)=\frac{1}{\sqrt{V}}e^{i\boldsymbol{p}_i\cdot\boldsymbol{x}_1}\alpha(1)\begin{Bmatrix}p(1)\\ n(1)\end{Bmatrix}\quad\text{or}\quad\frac{1}{\sqrt{V}}e^{i\boldsymbol{p}_i\cdot\boldsymbol{x}_1}\beta(1)\begin{Bmatrix}p(1)\\ n(1)\end{Bmatrix}$$

$$\text{etc.}\quad\text{for}\quad 1\leq i\leq N-1$$

and

$$\varphi_N(1)=\begin{cases}R(r)\,Y_l^l(\theta\phi)\,\alpha(1)\begin{Bmatrix}p(1)\\ n(1)\end{Bmatrix} & j=l+\frac{1}{2}\\ R(r)\left\{\sqrt{\frac{1}{2l+1}}\,Y_l^{l-1}(\theta\phi)\,\alpha(1)-\sqrt{\frac{2l}{2l+1}}\,Y_l^l(\theta\phi)\,\beta(1)\right\}\begin{Bmatrix}p(1)\\ n(1)\end{Bmatrix} & j=l-\frac{1}{2}.\end{cases}$$

In this model, $N-1$ is assumed to be even and both spin and isotopic spin are saturated. $R_l(r)$ in the above expression satisfies the following relation : $\int_0^\infty R_l(r)j_l(kr)r^2dr=c\delta(k-k_F)$, where k_F is a Fermi momentum. V corresponds to $r_0=1$ in unit $\mu=1$, where r_0 is (nuclear radius) $\times$ (nucleon number)$^{-1/3}$.

Since the wave function used here is merely an approximate one, we cannot hope to obtain the exact value, but at least it will give a correct order of magnitude since the Fermi gas model wave function proves to be a fairly good approximation, when nuclei are rather heavy and a reasonable cut-off length could be chosen.

In table we list the final results for proton (in nuclear magnetons) thus obtained.

(i) $j=l+\frac{1}{2}$

k_0 \ l	0	1	2	3	4	5
1	−0.12	−0.14	−0.13	−0.15	−0.15	−0.17
2	−0.14	−0.16	−0.18	−0.19	−0.23	−0.25
$3\to\infty$	−0.15	−0.15	−0.19	−0.22	−0.25	−0.29

(ii) $j=l-\frac{1}{2}$

k_0 \ l	1	2	3	4	5
1	0.02	0.03	0.05	0.04	0.04
2	0.00	−0.01	−0.03	−0.06	−0.07
$3\to\infty$	−0.01	−0.03	−0.07	−0.09	−0.12

where k_0 is the cut-off momentum (in unit $\mu=1$). Signs must be changed in these values in the case of neutrons.

§ 4. Concluding remarks

Our new approach to the problem of the exchange current effect on the nuclear magnetic moments has been shown to be very powerful in that the experimentally observed results have been adopted in a framework of our theoretical description. No detailed knowledge of the interaction mechanism is needed, only if the initial and final states could be exactly specified by observation.

The main reason for that we have chosen H^3 as one of the examples is that it was the first nucleus to indicate the possible effects of the exchange currents in it. The deuteron magnetic moment anomaly, on the other hand, can be explained without taking into account any of the exchange effects, that is by means of the admixture of the D state. But, this is not the case with triton moment. No reasonable admixture is enough to account for the anomaly of triton magnetic moments. Because of this fact, we attempted to explain this situation in terms of the exchange effects.

It turned out, however, that the whole exchange effects gave us the correction of about $\simeq+0.24$ n. m. to be added to the proton moment, and this amount of correction evidently exceeds the observed value by about 0.054 n. m. This deviation might possibly be brought to zero by the admixture of reasonable states.

As to the second example taken up in our calculations, a brief remark must be added. Although the shell model has succeeded in explaining various properties of atomic nuclei, the quantitative discussions of them using this model cannot yet be made with satisfaction. For instance, the deviation of the magnetic moment from the Schmidt lines still remains large. For this deviation many sorts of explanations have been given, some of them being the modifications of the shell model itself. However, even in such an

attempt it is important to notice how large the exchange magnetic moments are. These exchange moments of heavy nuclei were calculated by Miyazawa[7] and indicated the possibility of obtaining the large value enough to account for the deviation from the Schmidt lines. But, Miyazawa's calculations were performed using a very large coupling constant ($\gtrsim 0.25$) and neglecting the finite extension of the nucleons, and this fact would be the cause of the large values of his results. Our results, on the contrary, turned out to be small, so we must seek for some other way of the explanation thereof.

To conclude our discussions, the authors are very grateful for the guidance by and many useful discussions with Dr. Miyazawa.

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Proton-Proton Scattering and Pion Theory of Nuclear Forces*

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In order to explore reliable consequences of the pion theory of nuclear forces at high energies, the $p-p$ scattering at 90 Mev is discussed in detail. No evidence has been found to show that the static pion-theoretical potential is appreciably modified up to 100 Mev. This conclusion is in conflict with the prediction of *very strong* spin-orbit coupling potentials recently made by Signell and Marshak and by Gammel and Thaler. Especially, it is discussed that the former spinorbit term was introduced in reliance on Gartenhaus' potential at small inter-nucleon distances. These spin-orbit coupling potentials are unreasonably strong from the pion-theoretical view-point. Importance of the ${}^3P_2-{}^3F_2$ coupling parameter in determining angular distributions is pointed out.

§ 1. Introduction

In our previous works on the verification of the pion theory of nuclear forces,[1] it was shown that all experimental data at low energies (below 20 Mev) are quantitatively explained without any exception and that the explanation is made mainly in terms of the outer part (outside the pion Compton wave length) of the static pion-theoretical potential. The effective $\pi-N$ coupling constant was thereby determined as $g_e^2/4\pi=0.080\pm0.010$.

One of the essential points of our method of approach used there is dividing the inter-nucleon distance into three regions and distinguishing the reliability of the potential or the nuclear interaction in each region. It was also discussed that the validity of this method is necessarily limited to regions of not so high energy (below about 100 Mev), since: (i) Interactions at small distances, which cannot be defined by the present-day quantum theory, become important at the high energy. Especially angular distributions are seriously dependent on the inner interactions mainly through the P-wave phase shifts. (ii) Dynamical corrections to the outer part of the potential are expected to be appreciable and modify it. Properties of the corrections are directly related to properties of the pion-cloud and possibly to the nature of the nucleon core as the source of the cloud.

It may be said that we have had no serious attempts to explore reliable consequences of the pion theory of nuclear forces in the high energy region.** Works on nuclear forces around 100 Mev can be roughly divided into two groups. The first group[2] consists of works of phase shift analysis.[2] Unfortunately, these works seem to be performed under purely phenomenological and mathematical programs, utterly ignoring the expectation

* Some of preliminary results of this paper have already been published.[13]

** For references, see ref. 1, p. 71.

that some of characteristic features of the low energy pion theory should still remain unchanged to some extent at high energies. Moreover, we cannot expect the phase shift anlysis to give a unique solution easily. The second group includes studies of mechanical application of the static pion-theoretical potential, even to scattering at 300 Mev.[3] Since the reliability of the potential as discussed in detail by our group[1] is scarcely taken into account in these works, we can get rather few informations on the pion theory of nuclear forces from them.

In the present paper, consequences of the static pion-theoretical potential established at the low energies are compared in detail with the $p-p$ scattering experiments at 90 Mev. As will be discussed in § 2, D- and higher-angular-momentum waves (with the impact parameter larger than the pion Compton wave length) are given their phase shifts by the outer part of the pion-theoretical potential. Low-angular-momentum waves are, on the contrary, treated more or less phenomenologically as is usually done in the phase shift analysis. In the same way, the $n-p$ scattering is analysed by Watari in the following paper.[4] In our analyses in § 3, we shall find no evidence to show that the static pion-theoretical potential is modified appreciably. As will be discussed in § 4, any velocity-dependent interactions would be, if any, still weak around 100 Mev, and no drastic changes may be expected in the structure of the pion-cloud at such energies.

Analyses at higher energies ($\sim$150 Mev) have been done by Tamagaki.[5] It is to be noted that although his analyses are necessarily far more phenomenological than ours, he obtained the same conclusion on the velocity-dependent interactions. In § 5, remarks will be made on spin-orbit coupling potentials recently introduced by Signell and Marshak[6] and by Gammel and Thaler.[12]

§ 2. Phase shifts and the applicability of the pion-theoretical potential

According to the division of the inter-nucleon distances into three regions,[1] we classify phase shifts also into three categories:

I) *Phase shifts with* $L \geq 2$ are treated pion-theoretically, where $L\hbar$ is the angular momentum. Since the D-wave impact parameter $b_2 = 1.7$*, they are almost completely determined by the outer part of the one-pion-exchange potential in the region I ($x \gtrsim 1.5$). They are shown in Table 1a.

II) *P-wave phase shifts* with $b_1 = 1.0$ are under the influence of both the one- and the two-pion-exchange potentials. They are reliable as far as their qualitative features are concerned. Not only ^3P-wave phase shifts $^3\delta_0^1$, $^3\delta_1^1$ and $^3\delta_2^1$, but also ϵ_2 (the $^3P_2-{}^3F_2$ coupling) and $^3\delta_2^3$ (the eigen $^3F_2$-wave phase shift) fall into this category.

The phase shifts of this category would be most sensitive to various dynamical corrections to the potentials, since, if such corrections are important, they would be in the region II ($1.5 \gtrsim x \gtrsim 0.7$) where the pion cloud has not the classical nature as in

* Unit of length of the inter-nucleon distance is the pion Compton wave length (1.415×10^{-13} cm). Throughout this paper notations and terminologies are the same as those in reference 1.

Table 1. Phase shifts and coupling parameter with $L \geq 1$ at 90 Mev. They are calculated by assuming the Taketani-Machida-Onuma potential with $g_c^2/4\pi = 0.08$.[6a),6b)] This TMO potential is considered here as the standard since in the region I it has the correct asymptotic form, and in the region II it has all features that are common with various potential having the plausible pion-theoretical basis.

a) $L \geq 2$. Pion-theoretically quite reliable.

state	phase shift	value
$^1D_2$	$^1\delta_2$	0.047
$^3F_3$	$^3\delta_3^s$	-0.026
$^3F_4$	$^3\delta_4^s$	0.003_5

b) $L=1$. Only qualitatively reliable. The inner interaction is cut off by a hard core of the radius $x_c = 0.33$.

state	parameter	value
$^3P_0$	$^3\delta_0^1$	0.377
$^3P_1$	$^3\delta_1^1$	-0.187
$^3P_2+{}^3F_2$	$^3\delta_2^\alpha$	0.065
	$^3\delta_2^T$	-0.035
	ϵ_2 (coupling parameter)	-0.704*

* The sign for ϵ_2 was missed in the calculation of ref. 6b.

the region I and is relatively complicated. However, their features are strongly dependent on the cut-off momentum of the virtual pions and we cannot derive the corections on a sound basis.

At 100 Mev the phase shifts of this category are also appreciably dependent upon the inner interactions in the region III ($x \lesssim 0.7$) if these interactions were to be strongly attractive. However it is possible and reasonable, as the first approximation, to make the inner interactions of secondary importance for the phase shifts, up to about the excitation energy of the nucleon isobar, by means of some cut-off procedures in the coordinate space. As will be discussed later, it seems to us that undue reliance on the region III interactions of Gartenhaus' potential[8] made Signell and Marshak[9] introduce an unreasonably large spin-orbit coupling potential.

Aside from non-static potentials which are sufficiently small to be neglected in the region I, the static pion-theoretical potential has the following characteristic features in the triplet odd state as can be seen from Fig. 1:

The one-pion-exchange potential

+ (negative tensor and attractive central) two-pion-exchange potential.

It may deserve special attention that this potential predicts the ^3P-wave phase shifts such as

$$^3\delta_0^1 \gg 0, \quad ^3\delta_1^1 \ll 0, \quad ^3\delta_2^\alpha \gtrsim 0, \quad \text{and} \quad ^3\delta_0^1 \sim 2|^3\delta_1^1|. \tag{1}$$

These relations come from the fact that the *one-pion-exchange tensor potential* is much

stronger than the central potential as well as the two-pion-exchange potential, for in the Born approximation we can expect

$$^3\delta_0^1 \propto -(V_c - 4V_T),$$
$$^3\delta_1^1 \propto -(V_c + 2V_T), \quad (2)$$

and

$$^3\delta_2^\alpha \propto -(V_c - 0.4V_T)$$
$$+ (\text{contribution coming from the } {}^3F_2\text{-coupling}).$$

We shall call ^3P-wave phase shifts satisfying eq. (1) those of "pion type".

One example of the pion type phase shifts and the mixing ratio are given in Table 1b. The potential is cut off by a hard core with a radius $x \sim 0.3$.

III) *Singlet S-wave phase shift* $^1\delta_0$ is dependent upon the unknown interactions in the region III ($x \lesssim 0.7$). Therefore we should treat it phenomenologically and determine in such a way that the total cross section (or $d\sigma/d\Omega$) at 90 Mev may be reproduced.

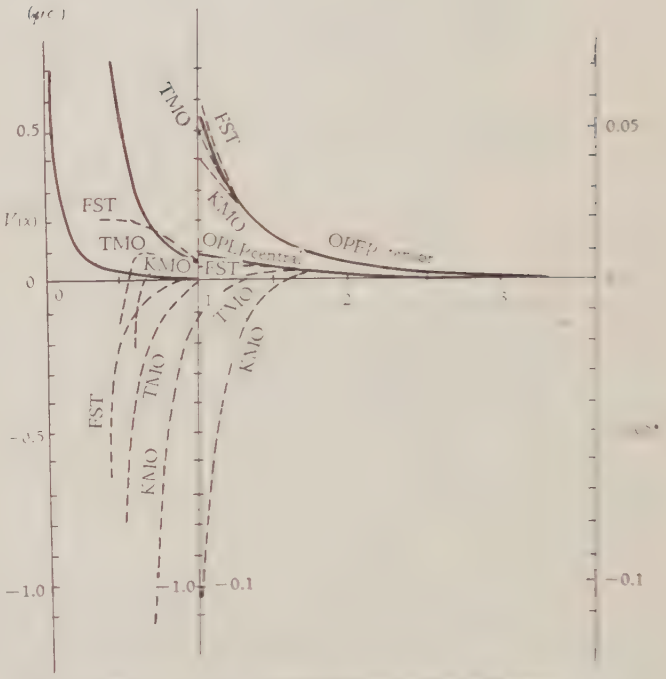

Fig. 1. Pion-theoretical potential in the triplet odd state
OPEP: The one-pion-exchange potential with $g_{ps}^2/4\pi = 0.08$.
TMO,[9a] FST,[10a] KMO[10b]: one-plus-two-pion exchange potentials.
For derivation methods refer reference [7], and its Fig IV on p. 11.

§ 3. Analysis of $p-p$ scattering at 90 Mev

When we assume the ^3P-wave phase shifts of the pion type, the angular distribution $d\sigma/d\Omega$ and the polarization cross section $P(\theta)d\sigma/d\Omega$ are effectively dependent only upon one parameter, the $^3P_2 - ^3F_2$ coupling ϵ_2. We would discuss the ϵ_2-dependence of $d\sigma/d\Omega$ and of $P(\theta)d\sigma/d\Omega$ and determine ϵ_2 rather phenomenologically. Reasonings for this prescription are: (i) The strong positive tensor force predicts eq. (1). (ii) Of the $L=1$ scattering parameters, only ϵ_2 is proportional to k^2 at low energies, while the others at least to k^3, which seems to mean that the energy dependence of ϵ_2 is complicated and may be sensitive to the inner interactions.

First, we will discuss the polarization cross section, which is shown in Fig. 2, since $P(\theta)$ is due to the tensor force and is very sensitive to ϵ_2. Main terms of $P(\theta)d\sigma/d\Omega$

can be expressed as follows :*

$$P(\theta)\,d\sigma/d\Omega \sim (6/k^2\kappa^2) f(\epsilon_2)\,\{[01|2\alpha]+[2\gamma|01]+(3/2)[11|2\alpha]+(3/2)[2\gamma|11]\} \times \sin\theta\cos\theta, \tag{3}$$

where $$[J\rho|J'\rho'] \equiv \sin {}^3\delta^{\rho}_{J} \sin {}^3\delta^{\rho'}_{J'} \sin({}^3\delta^{\rho}_{J} - {}^3\delta^{\rho'}_{J'}), \quad \rho = \alpha,\ \beta,\ \gamma,$$

(However, for convenience sake, we use ${}^3\delta^1_0$ and ${}^3\delta^1_1$ instead of ${}^3\delta^{\gamma}_0$ and ${}^3\delta^3_1$ as done in ref. 1.)

and

$$f(\epsilon) = \cos 2\epsilon + \frac{1}{2\sqrt{6}} \sin 2\epsilon.$$

It is very convenient that $P(\theta)\,d\sigma/d\Omega$ is separated into two parts : the ϵ_2-dependent part $f(\epsilon_2)$ as shown in Fig. 3 and the δ-dependent part in the parenthesis. When the ${}^3P$ wave phase shifts are of the pion-type as eq. (1), all terms of the δ-part in eq. (3) are positive. This coincidence of the sign, which is due to the strong positive tensor potential, is quite favourable to the large experimental $P(\theta)\,d\sigma/d\Omega$ and gives

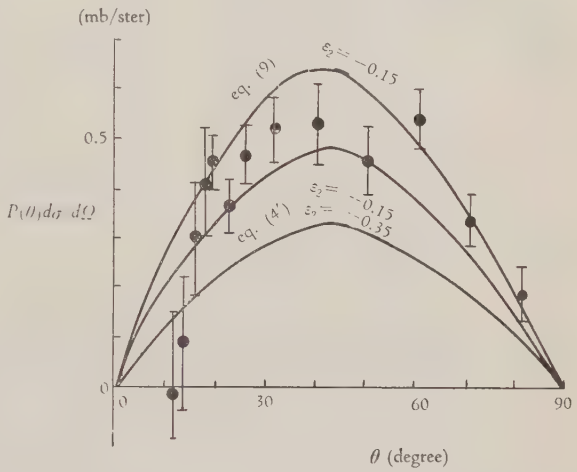

Fig. 2. Polarization cross section at 90 Mev with the phase shifts in Table 1 and eq. (9). ϵ_2 is varied. For the experimental data, see ref. 2.

$$P(\theta)\,d\sigma/d\Omega \sim f(\epsilon_2)\sin\theta\cos\theta \times 1.03\ \text{mb}. \tag{4}$$

Including the higher wave contributions as given in Table 1b, we have

$$P(\theta)\,d\sigma/d\Omega = f(\epsilon_2)\,(0.98+0.15\cos^2\theta)\sin\theta\cos\theta\ \text{mb}. \tag{4'}$$

By comparing eq. (4') with experimental data at 95 Mev[2]

$$P(\theta)\,d\sigma/d\Omega = \{(1.00\pm0.11)+(0.16\pm0.13)\cos^2\theta\}\sin\theta\cos\theta\ \text{mb}, \tag{5}$$

we have $f(\epsilon_2)\sim 1$. When we take the negative sign of ϵ_2 corresponding to Table 1b, we can estimate by Fig. 3

$$\epsilon_2(\text{pol}) \gtrsim -0.15. \tag{6}$$

It may be worth while to point out that $P(\theta)\,d\sigma/d\Omega$ is very sensitive to ϵ_2. Actually, for $\epsilon_2 = \epsilon_2(\text{TMO}) = -0.704$ in Table 1b, $P(\theta)\,d\sigma/d\Omega$ is vanishingly small owing to the accidental coinsidence with one of the roots of $f(\epsilon_2)$.

We shall then discuss the angular distribution $d\sigma/d\Omega$:

$$d\sigma/d\Omega = (1/k^2\kappa^2)\sum_L B_L P_L(\cos\theta), \tag{7}$$

* For complete formula, see reference 5).

where $B_0=\sum_{J,p}(2J+1)(\sin\delta^p_J)^2$ is independent of ϵ_2, while the other B_L's are not. Main terms of B_2 are shown in Table 2.*

As shown in Fig. 4, the angular distribution becomes unisotropic as ϵ_2 increases. This is due to the following situations. As can be seen in Table 2, all main terms of B_2, the coefficient of $P_2(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1)$, are non-positive at $\epsilon_2=\epsilon_2(\mathrm{TMO})$, while they become non-negative without any exception as ϵ_2 increases. Therefore, as long as the ^3P-wave phase shifts of the pion type are assumed, the angular distribution becomes more and more unisotropic as ϵ_2 increases. We can estimate therefore

$$\epsilon_2(\text{a. d.})\lesssim -0.35. \qquad (8)$$

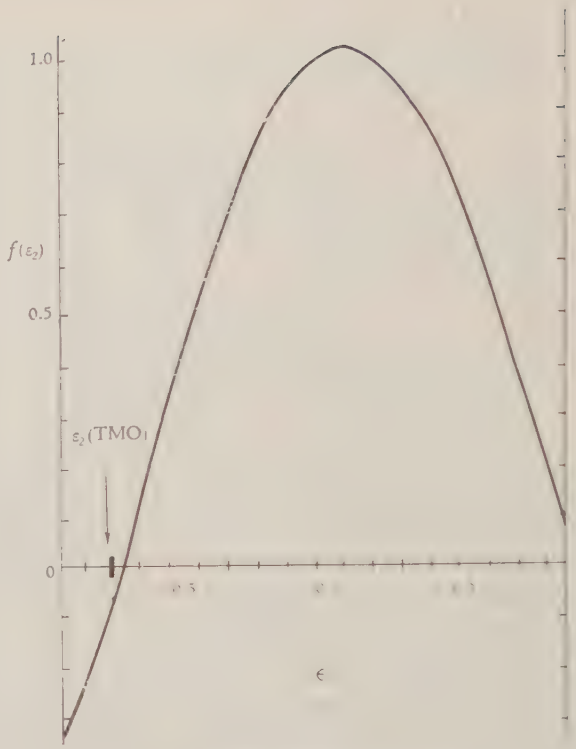

Fig. 3. $f(\epsilon_2)$ defined by eq. (3).

The small inconsistency between $\epsilon_2(\text{pol})\gtrsim -0.15$ and $\epsilon_2(\text{a. d.})\lesssim -0.35$ is, however, not serious at all. The δ-dependent part of $P(\theta)d\sigma\ d\Omega$ is sensitive to the magnitudes of the ^3P-wave phase shifts (proportional to δ^3 for small δ). Therefore if δ increase a little, $d\sigma\ d\Omega$ and $P(\theta)d\sigma\ d\Omega$ can be reproduced with one value of ϵ_2 consistently. An example is as follows. Let Δ denote a variation of the contribution from the tensor

Table 2. Main triplet terms of B_2 and their ϵ_2-dependence. Notice that as ϵ_2 increases, none of them decreases.

$$B_2=\sum_{J\rho J'\rho'} b_{J\rho J'\rho'}(\epsilon_2)\,(J\rho|J'\rho'),$$

where

$$(J\rho|J'\rho')=\sin{}^3\delta^\rho_J\sin{}^3\delta^{\rho'}_{J'}\cos({}^3\delta^\rho_J-{}^3\delta^{\rho'}_{J'}).$$

$J\rho$	$J'\rho'$	$(J\rho\|J'\rho')$	$b_{J\rho J'\rho'}(\epsilon_2)$	sign of $b_{J\rho J'\rho'}$ at $\epsilon_2=-0.7,\ 0,\ +0.7$
11	11	0.035	3/2	a positive constant
01	2α	0.023	$4[\cos\epsilon_2+\sqrt{3/2}\sin\epsilon_2]^2$	$\sim 0<+<+$
01	2γ	−0.012	$4[-\sin\epsilon_2+\sqrt{3/2}\cos\epsilon_2]^2$	$+>+>\sim 0$
11	2α	−0.012	$9[\cos\epsilon_2-\sqrt{2/3}\sin\epsilon_2]^2$	$+>+>\sim 0$
11	2γ	0.007	$9[\sin\epsilon_2+\sqrt{2/3}\cos\epsilon_2]^2$	$\sim 0<+>\sim +$

* The complete expression is given in reference 5).

potential to the ^3P-wave phase shifts. In eq. (2) let us assume that $\Delta = 0.01$ which corresponds to about 10% increase of the tensor contribution of the TMO potential. Such uncertainty of 10% may come from the uncertainty of the coupling constant or from the two-pion-exchange potential whose precise shape is not known as shown in Fig. 1. According to eq. (2), Δ adds 4Δ to $^3\delta_0^1$, -2Δ to $^3\delta_1^1$ and 0.4Δ to $^3\delta_2^\alpha$, and we have

$$^3\delta_0^1 = 0.42,\ ^3\delta_1^1 = -0.21$$
$$\text{and } ^3\delta_2^\alpha = 0.07. \qquad (9)$$

This example of the ^3P-wave phase shifts gives precise fits to $d\sigma/d\Omega$ and $P(\theta)d\sigma/d\Omega$ as shown in Figs. 2 and 5, and for $\epsilon_2 \sim -0.15$.

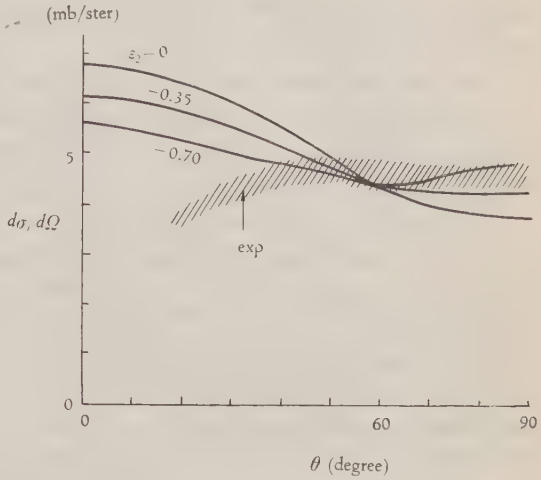

Fig. 4. Scattering cross section at 90 Mev with the phase shifts in Table 1. ϵ_2 is varied. $^1\delta_0$ is assumed as 0.50. For the experimental data, see ref. 2.

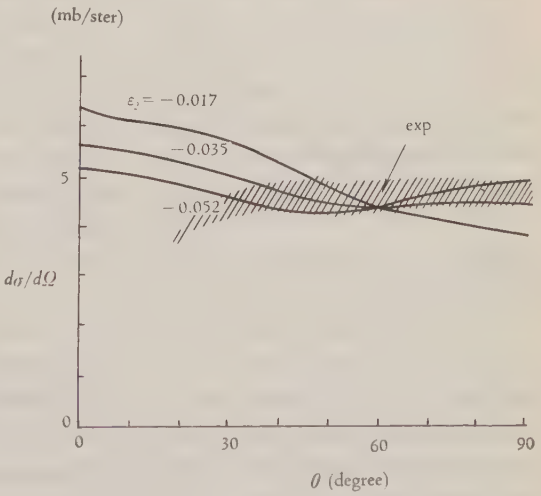

Fig. 5. Scattering cross section at 90 Mev with the shifts in eq. (9). ϵ_2 is varied. $^1\delta_0$ is assumed as 0.40. For the experimental data, see ref. 2).

§ 4. Concluding remarks

i) *Conclusions*

In § 3, it was shown that $p-p$ scatterings at 90 Mev, and consequently those below 100 Mev,* can be reproduced by the outer part of the static pion-theoretical potential, if the inner interactions in the triplet odd state are made non-essential by a hard-core cut-off. Watari also obtained the same conclusion in his analysis of $n-p$ scatterings at 90 Mev,[4] that the higher wave phase shifts, which are pion-theoretically reliable and classified under the categories I and II, are not contradictory to the experimental data. Namely, we find no evidences to show that the static pion-theoretical potential is appreciably modified below 100 Mev. No drastic change may be expected in the structure of the pion cloud. Our conclusion is conflict with predictions of *strong* spin-orbit coupling

* A complete fit to $p-p$ angular distribution at 18.2 Mev has already been obtained.[1]

potentials recently made by Gammel and Thaler,[12] and by Signell and Marshak.[9] This point will be discussed below†.

ii) *Remarks on Gammel and Thaler's spin-orbit coupling potential*[12]*

Gammel and Thaler discussed about an evidence upon the spin-orbit coupling force making use of Stapp's results of the phase shift analysis of 310 Mev $p-p$ scattering. They proposed a potential which has the following qualitative features:

(a) a very weak central potential.

(b) a positive tensor potential with a long range (range$\sim$0.9).

(c) a negative spin-orbit potential with a short range (range$\sim$0.2).

In the features (a) and (b) Gammel and Thaler's potential is similar to the pion-theoretical potential. Hwever, above as low as a few tens Mev, Gammel and Thaler's phase shifts cease to be of the pion type. For example, as shown in Table 3, they do not satisfy the inequality (1) but

$$^3\delta_2^x \sim {}^3\delta_0^1 \gg 0 \gg {}^3\delta_1^1.$$

This deviation from the pion type is due to their strong spin-orbit potential:

$$V_{LS}^{(GT)} = V_0 e^{-ax}/ax,$$

$$V_0 = -7122.5 \text{ MeV},$$

$$a^{-1} = 0.19.$$

Although the range is small, its depth is exceedingly large. Actually, the spin-orbit potential is much larger than the pion-theoretical tensor potential inside $x \sim 0.9$. Therefore, Gammel and Thaler's potential gives fits to the experimental data in quite a different way from ours. In view of the success of the low energy pion theory, it would not be reasonable to suppose that the pion-theoretical aspects of nuclear forces may be lost even at as low as a few tens Mev.

iii) *Remarks on Signell and Marshak's spin-orbit coupling potential*[9]

Recently Signell and Marshak also concluded the necessity of a spin-orbit coupling potential, which is added to Gartenhaus' potential,[7] by analysing the $p-p$ scattering data at the energy region around 150 Mev. We think that the disagreement between their results and ours comes from different ways of thinking about the inner interactions and, consequently, different methods of the cut-off in the triplet odd state as will be explained below.

As discussed in detail by Japanese physicists, the inner interactions in the region III is beyond the applicability of the present-day pion theory and should be treated phenomenologically.[1] Availing ourselves of this point of view Gartenhaus' potential was commented already in 1956.[1,7] However, Signell and Marshak state that "there is no free parameters in Gartenhaus' potential," and do not give proper phenomenological prescription to it. It is interesting to compare phase shifts by Gartenhaus' potential and those by

† See also note added in proof.

* The author wishes to thank Dr. Gammel and Dr. Thaler for sending him their preprint.

Table 3. Phase shifts and coupling parameter with $L=1$ by various potentials.

		${}^3\delta_0^1$	${}^3\delta_1^1$	${}^3\delta_2^\alpha$	${}^3\delta_2^\gamma$	ϵ_2
Gammel-Thaler,[12]	80 Mev	0.188	−0.223	0.168	0.002_5	0.326
	100 Mev	0.153	−0.251	0.196	0.008_5	0.291
Gartenhaus[8),3)]	90 Mev	0.93	−0.12	0.21	−0.06	0.23
Signell-Marshak[9)]	100 Mev	0.276	−0.244	0.171	−0.026	−0.258
Ours (Table 1b)	90 Mev	0.377	−0.187	0.065	−0.035	$\sim\begin{cases}-0.15\\-0.35\end{cases}$
(eq. (9))	90 Mev	0.42	−0.21	0.07	−0.04	−0.15

Signell and Marshak's one tabulated in Table 3. Large positive value ${}^3\delta_0^1$ and small magnitude of ${}^3\delta_1^1$ of the former are due to the very deep inner central potential (-1.5 Bev at $x=0$), and cannot explain the $p-p$ scattering data. From Table 3 we can immediately see that the effects of this inner potential, which is absolutely meaningless from the pion-theoretical point of view, are largely cancelled by the spin-orbit term and that the resultant phase shifts are not similar to Gammel and Thaler's but rather to ours. It should be pointed out that, contrary to Signell and Marshak's discussions, Gammel and Thaler's spin-orbit potential reproduces the experimental data in quite a different way from Signell and Marshak's, as shown in Table 3, especially, in different signs of ϵ_2.

The cut-off that Signell and Marshak adopted in their calculation is the zero cut-off at the nucleon Compton wave length. If a phenomenological cut-off by a hard core were assumed throughout, the results would be much changed, since the outer part of Gartenhaus' potential is quite the same with ours. Therefore, we could conclude that it is the undue reliance on the inner interactions that necessitates Signell and Marshak to introduce such a large spin-orbit potential.

It is interesting that the same conclusion is obtained by Tamagaki in his analyses of the 150 Mev data.[5)]

iv) *Remarks on the phase shift analysis*

In most of the works of the phase shift analysis,[7)] the ${}^3P_2-{}^3F_2$ coupling is completely neglected for the sake of simplicity, namely it is assumed that $\epsilon_2=0$.

Our analysis in § 3, however, shows that both the scattering cross section and the polarization are very sensitive to ϵ_2. One could hardly claim that results of the phase shift analysis are correct, if $\epsilon_2=0$ is assumed identically. Moreover, the presence of the strong tensor force of the pion-theoretical potential does not justify neglecting the ${}^3P_2-{}^3F_2$ coupling at all. Simplicity of analysis should not be preferred to the reality of the pion theory. Rather, correlations between ${}^3\delta_2^\alpha$, ${}^3\delta_2^\gamma$ and ϵ_2 should be investigated under the assumption of such a strong positive tensor potential as the pion-theoretical one.

The author would like to express his sincere thanks to Dr. R. Tamagaki and Dr. W. Watari, whose stimulating discussions have helped him develop this work.

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Noce added in proof. Recently Sato calculated the spin-orbit coupling potential making use of an extended method of ref. 10b, i.e., taking into account the features of the pion clouds somewhat phenomenologically*. His result is almost the same with the previous one**, which was calculated by the Fukuda-Sawada-Taketani formalism[10a] treating the normalization condition correctly for the Tamm-Dancoff amplitude. The latter potential is:

$$V_{LS} = -(g^2/4\pi)^2(2\mu^2/M)\{3-2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\}$$
$$\times(1+2/x+2/x^2+1/x^3)\exp(-2x)\,(\boldsymbol{\sigma}\times\boldsymbol{\nabla})\cdot\boldsymbol{\sigma}.$$

These pion-theoretical spin-orbit coupling potentials are by far weaker than the phenomenological ones, *especially in the outer region*, as shown below:

Magnitude of V_{LS} (aside the negative sign, in unit of $0.01\mu c^2$, μ the pion mass)

	$x=1$	$x=2$
pion-theoretical one	0.77	0.0058
Gammel-Thaler	5.4	0.019
Signell-Marshak	5.7	0.30

Thes fact supports our conclusion that any velocity-dependent interactions are not strong at least around 100 Mev.

* A private communication from S. Sato.

** See ref. 7, p. 126.

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Neutron-Proton Scattering and Pion Theory of Nuclear Forces*

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The n-p scattering at 90 MeV is discussed in detail from the standpoint of reference 1). Around this energy phase shifts with $L \geqslant 2$ are almost completely determined by the one-pion-exchange potential. Phase shifts with $L=1$ should be qualitatively determined by the pion theory. Phase shifts with $L=0$ must be determined phenomenologically. In this paper, using P-wave phase shifts obtained by Otsuki from the p-p scattering data, we determine scattering parameters in the $(J=1)$-states from the n-p scattering analysis. In section 2, n-p polarizations are analyzed and some sets of phase shifts with $J=1$ which are consistent with the experiments are determined. Using thus determined phase shifts, angular distributions are calculated and compared with experimental data in section 3. Thus we find that the static pion-theoretical potential can reproduce n-p experimental data at 90 MeV without any serious modifications.

§ 1. Introduction and summary

In 1956, we discussed low energy properties of nuclear forces in detail from the stand-point of the pion theory and found that all of them are quantitatively explained in terms of the static pion-theoretical potential with the coupling constant $g_e^2/4\pi = 0.080 \pm 0.010$.[1)]** This success is a motivation of the present work, where it is discussed that the pion-theoretical potential established at low energies can reproduce n-p scattering at 90 MeV. At 90 MeV, we have two kinds of useful data, namely, single scattering cross section $d\sigma/d\Omega$ over all angles[2)] and polarization cross section $P(\theta)\, d\sigma/d\Omega$[3)]. We already calculated the former in 1953[4)] and the latter in 1956[5)], assuming a static pion theoretical potential derived by the perturbation approximation up to the fourth order (the TMO potential). The agreement between the experiment and the theory was fairly satisfactory. However, the success of the low energy pion theory, seems to require more detailed investigations for high energy phenomena; especially we must find out which features of the potential bring about the agreement and what modifications of them make the agreement better.

Around 90 MeV, corrections to the static pion-theoretical potential, e.g. the relativistic effects, the recoil effects, etc., become appreciable. Unfortunately we have few knowledge about these corrections. But they are expected to be still small in the region

* The preliminary report of this work has been published in this journal[9)].

** Notations and terminologies used in this paper are the same with those in ref. 1).

I. Even in the region II they may not be so large as to mask main features of the static potentials in this region. Therefore it is desirable to investigate properties of these corrections phenomenologically by analyzing experimental data with the static pion-theoretical potentials established by low energy analyses.

On the other hand, at the energy of 90 MeV, the impact parameter of P-wave is $b_1 \approx 1.0$ and so the angular distributions are not seriously affected by the two-nucleon interactions at small distances[1]. The analysis from the standpoint of the pion theory developed in reference 1) is still meaningful in the problem of nuclear forces.

The p-p scattering has been analyzed by Otsuki[2], and his results are referred to in our analysis.

We shall divide phase shifts into three classes as was done in the p-p analysis.

(I) Phase shifts with $L \geq 2$. Impact parameters of the wave of $L \geq 2$ are larger than 1.7. They are almost completely determined by the one-pion-exchange potential in the region I. Their values are tabulated in Table I. These values are reliable within the uncertainty of several per cent due to the ambiguities of the coupling coustant and the inner region nuclear forces.

(II) Phase shifts with $L=1$. P-wave impact parameter is $b_1 \approx 1.0$, and their phase shifts are mainly determined by the potential in the regions II and I. Therefore their qualitative features can be regarded as pion-theoretical.

(III) Phase shifts with $L=0$. S-wave is much influenced by the interactions in the region III. So we can say nothing about S-wave phase shifts from the present-day pion theory. They must be determined phenomenologically as will be done below. Their magnitude and energy dependence may reflect some features of the interactions in region III.

We analyze the n-p polarization in section 2, and determine phase shifts with $L=1$ and 0 which are consistent with the experiment. Using thus determined phase shifts, n-p angular distribution is calculated and compared with the experimental data in section 3.

Our conclusions are as follows:

(i) According to the method of approach proposed in reference 1), the static pion-theoretical potential established at low energies can reproduce n-p experimental data at 90 MeV *without any serious modification*. Considering Otsuki's result of the p-p analysis, we could say that the pion theory of nuclear forces explains all experimental data up to 100 MeV correctly.

(ii) In the triplet odd state, however, the tensor potential seems to be more or less stronger than that of the TMO potential. This result is consistent with a disposition of the FST potential.

Table I. $L \geqq 2$ phase shifts in radians determined by the pion theoretical potential

Triplet				Singlet		
$^3D_2$	$^3D_3$	$^3F_3$	$^3F_4$	$^1D_2$	$^1F_3$	$^1G_4$
0.29	0.00	−0.03	0.00	0.05	−0.04	0.00

(iii) We find some evidences to show that in both the triplet and singlet even states, hard-core-like repulsive interactions are present at small distances.

§ 2. *n-p* polarization

The polarization cross section $P(\theta)\,d\sigma/d\Omega$ is analyzed firstly by the following reasons.
(i) Phase shifts of the singlet states do not contribute to this phenomena.
(ii) $P(\theta)\,d\sigma/d\Omega$ includes phase shifts in a form $[\delta_A|\delta_B]=\sin\delta_A\cdot\sin\delta_B\cdot\sin(\delta_A-\delta_B)$. Therefore higher wave phase shifts are safely neglected since $[\delta_A|\delta_B]\sim\delta^3$ for $|\delta_A|\sim|\delta_B|\sim\delta\ll 1$.

The *n-p* polarization cross section is expressed in terms of $P_L(\cos\theta)$ as follows:

$$P(\theta)\;d\sigma/d\Omega=k^{-2}\sin\theta\sum_L a_L\,P_L(\cos\theta).$$

Experimental results by Stafford *et al.*[3] are as follows and are shown in Fig. 1.

$$P(\theta)\,d\sigma/d\Omega=\sin\theta[(1.17\pm0.11)P_0+(2.34\pm0.23)P_1+(0.21\pm0.26)P_2]\ \text{mb/sr.}$$

(for 95MeV)

$$\approx\sin\theta[(1.12\pm0.09)P_0+(2.28\pm0.21)P_1]\ \text{mb/sr.}$$

When the phase shifts by the TMO potential shown in Table II-a are assumed[5]* we have the following result:

$$P(\theta)\,d\sigma/d\Omega=\sin\theta[0.61P_0+2.14P_1+0.03P_2-0.02P_3]\ \text{mb/sr.}$$

Signs of the main terms, a_0 and a_1, coincide with the experimental ones. However, since their magnitudes are a little smaller, we would discuss this point more in detail below.

Main parts of a_0 and a_1, which we denote as $(a_0)_0$ and $(a_1)_0$, are as follows:**

Main parts of a_0 (due to interferences between different parity states)

$$\left.\begin{aligned}
&{}^3P_0-({}^3S_1+{}^3D_1): && \tfrac{1}{2}g(\epsilon_1)\{[0\gamma|1\alpha]-[0\gamma|1\gamma]\},\\
&{}^3P_1-({}^3S_1+{}^3D_1): && (3/4)\,g(\epsilon_1)\{[1\beta|1\alpha]-[1\beta|1\gamma]\},\\
&({}^3P_2+{}^3F_2)-({}^3S_1+{}^3D_1): && F(\epsilon_1,\epsilon_2)[1\alpha|2\alpha]+F(\epsilon_1,\epsilon_2+\pi/2)[1\alpha|2\gamma]\\
&&& +F(\epsilon_1+\pi/2,\epsilon_2)[1\gamma|2\alpha]+F(\epsilon_1+\pi/2,\epsilon_2+\pi/2)[1\gamma|2\gamma],
\end{aligned}\right\}(a_0)_0$$

where

$$[J\rho|J'\rho']=\sin{}^3\delta_J{}^\rho\,\sin{}^3\delta_{J'}{}^{\rho'}\sin({}^3\delta_J{}^\rho-{}^3\delta_{J'}{}^{\rho'}),$$

$$g(\epsilon_1)=(3/2\sqrt{2})\sin(2\epsilon_1+\omega),\quad\tan\omega=2\sqrt{2},$$

---

* The first term of the last equation in page 1204 of reference 5) is not correct. The revised equation is given here. The equation in page 1206 should also read as $P(d\sigma/d\Omega)=\sin\theta\times1.5P_1(\cos\theta)$ mb/sr. Consequently, curves of Fig. 1 of reference 5) are not correct. The first two sentences (from "However" to "reasons.") of the second paragraph in page 1205 must be deleted.

** Complete expressions of coefficients a_L are given in a forthcoming paper by R. Tamagaki[7].

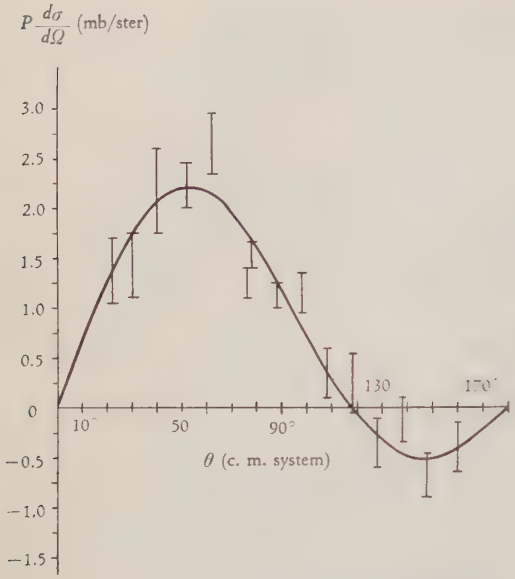

Fig. 1 *n-p* polarization cross section. Experimental data are those of ref. 3). The curve is calculated using phase shifts given in page 9.

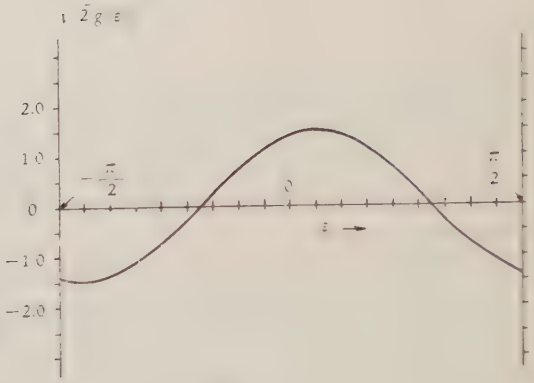

Fig. 2

and

$$F(\epsilon_1, \epsilon_2) = \cos^2\epsilon_2 + \tfrac{1}{4}\cos^2\epsilon_1\cos^2\epsilon_2 + (5/8\sqrt{2})\sin 2\epsilon_1(\cos^2\epsilon_2 - (2/3)\sin^2\epsilon_2) - (5/6)\sin^2\epsilon_2 - (7/12)\sin^2\epsilon_1\sin^2\epsilon_2 + (9/8\sqrt{6})\sin^2\epsilon_1\sin 2\epsilon_2.$$

$\sqrt{2}\, g(\epsilon)$ is plotted in Fig. 2.

Main parts of a_1 (due to interfences between same parity states)

$$\left.\begin{aligned} &({}^3S_1+{}^3D_1)-({}^3S_1+{}^3D_1): && -(9/4)g(\epsilon_1)[1\alpha|1\gamma], \\ &({}^3S_1+{}^3D_1)-{}^3D_2: && (5/4)g(\epsilon_1)\{[2\beta|1\alpha]-[2\beta|1\gamma]\}. \end{aligned}\right\} \quad (a_1)$$

As was discussed in detail in the *p-p* analysis[5] the triplet odd state phase shifts should rather be restricted in our analysis. Since the pion-theoretical potential has a strong positive tensor part and a weak central part, the phase shifts should be as follows:

$${}^3\delta_0{}^1 \geqslant 0,\ {}^3\delta_1{}^1 \leqslant 0,\ {}^3\delta_2{}^\alpha \gtrless 0,\ \text{and}\ {}^3\delta_0{}^1 \sim 2|{}^3\delta_1{}^1|.$$

Their values, which are calculated with the TMO potential (set *a*) and which are modified in the *p-p* analysis (set *b*), are tabulated in Table II. Table II contains also some sets of phase shifts which are taken in the present analysis.

We should like to determine phenomenologically the ${}^3S_1 + {}^3D_1$ state scattering parameters, ${}^3\delta_1{}^\alpha$, ${}^3\delta_1{}^\gamma$ and ϵ_1. For the moment, we take values of the set *b* for the triplet odd state parameters. By virtue of the continuity of the phase shift from its low energy value, ${}^3\delta_1{}^\alpha > 0$ and ${}^3\delta_1{}^\gamma < 0$. Then in order to reproduce $(a_1)_0 > 0$ we have $g(\epsilon_1) > 0$. There is another possibility in which $g(\epsilon_1) < 0$ and ${}^3\delta_1{}^\alpha$, ${}^3\delta_1{}^\gamma > 0$ or ${}^3\delta_1{}^\alpha$, ${}^3\delta_1{}^\gamma < 0$. However,

these phase shifts do not continue smoothly to the low energy values.

The phase shifts must satisfy the following conditions:

(i) In order to reproduce the magnitude of a_1, we have

$$(a_1)_0 = 0.28 - 0.21. \qquad \text{(for 95 MeV)}$$

Higher wave contribution which is mainly due to the ^3P-waves is estimated from the p-p polarization data as 0.02—0.03.

(ii) Moreover, because of $\sqrt{2}\, g(\epsilon_1) \leq 1.5$, we have

$$\frac{2}{g(\epsilon_1)}(a_1)_0 > 0.39. \qquad \text{(for 95 MeV)}$$

This quantity depends only on $^3\delta_1^{\alpha}$ and $^3\delta_1^{\tau}$.

(iii) To reproduce the magnitude of a_0, we have

$$(a_0)_0 = 0.146 - 0.116.$$

Higher wave contributions are estimated using values of phase shifts listed in Table I and Table II, set b, and are found to be about 0.01.

(iv) In order to reproduce the n-p total cross section, we have

$$(\sin {}^3\delta_1^{\alpha})^2 + (\sin {}^3\delta_1^{\tau})^2 = 0.5 \pm 0.1.$$

The higher wave contributions, which are estimated from the p-p cross sections and Table I, are subtracted from the experimental value.

Table II. $T=1$ phase shifts (in radians)

	$^3P_0$	$^3P_1$	$^3P_2$	$^3F_2$	ϵ_2	$^1S_0$
a(TMO)	0.377	−0.187	0.065	−0.035	−0.704	0.825
b	0.42	−0.21	0.07	−0.04	−0.26	0.40
c	0.45	−0.22	0.07	−0.04	−0.35	0.36
d	0.52	−0.26	0.08	−0.04	−0.18	0.00
d′	0.52	−0.26	0.08	−0.04	−0.35	0.27

First, we determine regions of $^3\delta_1^{\alpha}$ and $^3\delta_1^{\tau}$ which satisfy conditions (ii) and (iv). Then, by the condition (i), values of $g(\epsilon_1)$, and therefore values of ϵ_1 are determined* for each curve of $(2/g(\epsilon_1))(a_1)_0 = \text{const.}$ These procedures are graphically shown in Fig. 3-a. Next, for each set of $(^3\delta_1^{\alpha}, {}^3\delta_1^{\tau}; \epsilon_1)$, $(a_0)_0$ is calculated, and compared with the condition (iii). Thus the regions of $(^3\delta_1^{\alpha}, {}^3\delta_1^{\tau}; \epsilon_1)$ which are compatible with the n-p polarization experiment are determined. The shaded portions of the graph in Fig. 3-b are such regions of $(^3\delta_1^{\alpha}, {}^3\delta_1^{\tau}; \epsilon_1)$.

In the previous calculation[5] we took $^3\delta_1^{\alpha} = 0.52$, $^3\delta_1^{\tau} = -0.28$ and $\epsilon_1 = 0.1$. This point is outside the allowed regions.

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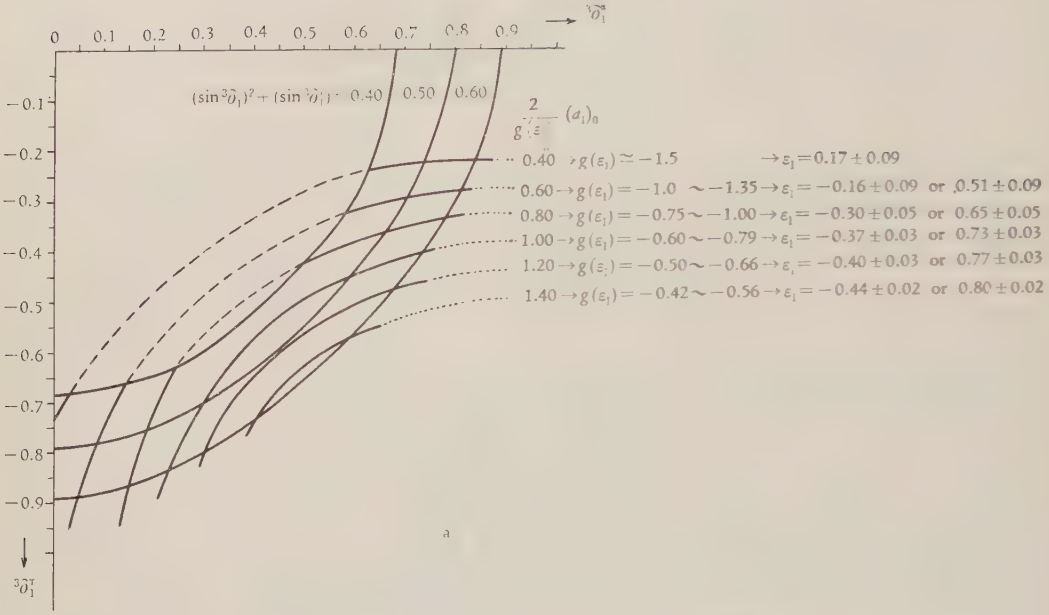
* Corresponding to one value of $g(\epsilon_1)$, there exist, in general, two values of ϵ_1 among which the larger value is denoted as $\epsilon_>$ and the smaller $\epsilon_<$.


0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9
$(\sin^2\delta_1)^2+(\sin^2\delta_1)^2=0.40$ 0.50 0.60
$\frac{2}{g^2}(a_1)_0$
$\rightarrow g(\varepsilon_1)\simeq -1.5 \rightarrow \varepsilon_1=0.17\pm0.09$
$0.60\rightarrow g(\varepsilon_1)=-1.0 \sim -1.35\rightarrow \varepsilon_1=-0.16\pm0.09$ or 0.51 ± 0.09
$0.80\rightarrow g(\varepsilon_1)=-0.75\sim -1.00\rightarrow \varepsilon_1=-0.30\pm0.05$ or 0.65 ± 0.05
$1.00\rightarrow g(\varepsilon_1)=-0.60\sim -0.79\rightarrow \varepsilon_1=-0.37\pm0.03$ or 0.73 ± 0.03
$1.20\rightarrow g(\varepsilon_1)=-0.50\sim -0.66\rightarrow \varepsilon_1=-0.40\pm0.03$ or 0.77 ± 0.03
$1.40\rightarrow g(\varepsilon_1)=-0.42\sim -0.56\rightarrow \varepsilon_1=-0.44\pm0.02$ or 0.80 ± 0.02
$^3\delta_1$
0 0.2 0.4 0.6 0.8 1.0
A B C
D F
$(a_0)_0=0.12$; (ε_1)
$(a_0)_0=0.12$; (ε_1)
$(a_0)_0=0.11$; (ε_1)
$^1\delta_1$

(b)

Fig. 3

§ 3. *n-p* angular distribution

In the previous section, we determined the regions of the phase shifts with $J=1$,

which can reproduce the *n-p* polarization data. In the present section, we shall investigate whether these phase shifts can reproduce the *n-p* angular distribution.

The angular distribution $d\sigma/d\Omega$ is expressed as

$$\frac{d\sigma}{d\Omega}=\frac{1}{4k^2}\sum_L B_L\, P_L(\cos\theta).$$

A remarkable feature of the experimental data is that $d\sigma/d\Omega$ is V-shape and is symmetric about 80° in the centre of mass system, and so

$$B_1,\ B_3\sim 0,\ \text{and}\ B_2\gg 0.$$

We have two allowed regions of the phase shifts determined in Fig. 3-b. In one of them, where $|^3\delta_1^{\alpha}|<|^3\delta_1^{\tau}|$, B_2 is always negative in disagreement with experiment. In the other, $|^3\delta_1^{\alpha}|>|^3\delta_1^{\tau}|$, we take several sets of phase shifts as typical examples which are shown in Fig. 3-b and Table III as Ⓐ—Ⓕ.

Table III. *n-p* angular eistributions

	$^3\delta_1^{\alpha}$	$^3\delta_1^{\tau}$	ϵ_1	$^1\delta_1^1$	B_0	B_1	B_2	B_3	B_4
Ⓐ	0.63	−0.23	0.09 0.26	−0.26	5.31	−1.62 −0.61	2.50 1.76	−1.34 −1.30	1.76
Ⓑ	0.74	−0.23	0.09 0.26	−0.20	5.86	−1.09 −0.04	2.47 1.88	−1.34 −1.26	1.65
Ⓒ	0.83	−0.23	0.09 0.26	−0.17	6.41	−0.78 +0.30	2.56 2.10	−1.24 −1.18	1.58
Ⓓ	0.74	−0.28	−0.07 0.42	−0.17	5.98	−1.98 +1.12	3.81 1.97	−1.24 −1.09	1.58
Ⓔ	0.67	−0.35	−0.31 0.63	−0.20	5.87	−3.66 +1.74	6.70 2.10	−0.99 −0.82	1.64
Ⓕ	0.74	−0.40	−0.35 0.70	−0.17	6.50	−3.87 +2.50	8.55 3.71	−0.94 −0.66	1.58

Assuming $T=1$ phase shifts in Table II-*b* and $L\geq 2$ phase shifts in Table I, the angular distribution is calculated and the results are given in Table III. Some curves are plotted in Fig. 4.

Since the $^1P_1$-wave phase shift is fairly dependent upon two-pion-exchange potential in the region II, its magnitude has some ambiguities within −0.25——−0.15, and determined by the condition to reproduce the total experimental cross section.

For our phase shifts B_4 is rather large, and gives almost flat angular distribution between 60° and 120°. This aspect also appears in the experimental data. B_3 is always small and negative. These general trends of B_3 and B_4 are almost determined by the higher wave phase shifts and should be regarded as one of the characteristic consequences of the one-pion-exchange potential in the region I.

B_2 and B_1 are sensitive to the mixing parameter ϵ_1. If ϵ_1 becomes smaller B_2

becomes larger and the hump of the angular distribution curve at about 90° is suppressed. At the same time, $-B_1$ becomes large and gives rise to a large backward scattering cross section.

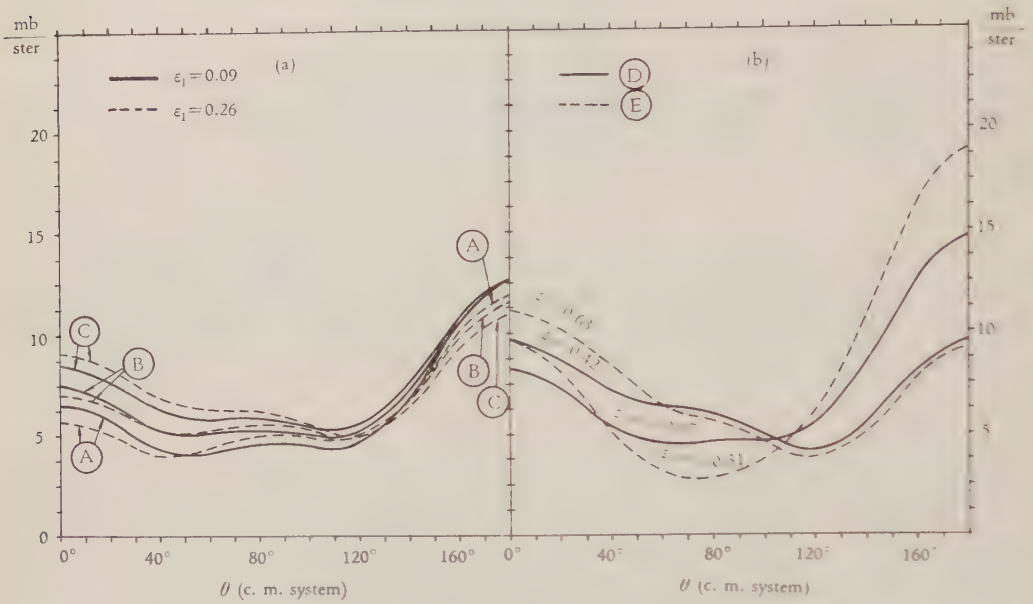
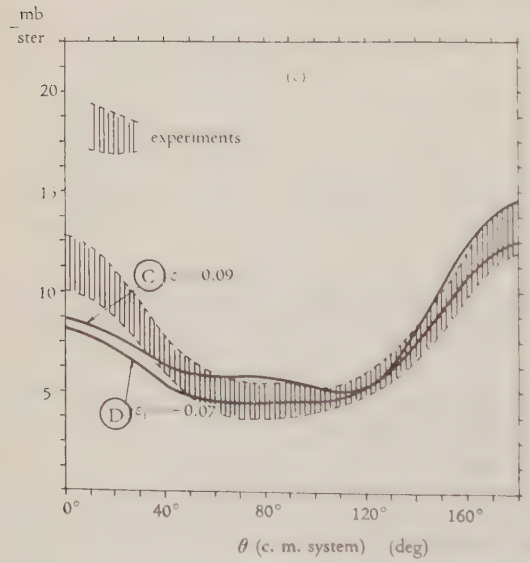


Fig. 4 n-p angular distributions. Some curves of Table III are plotted.



To restrict the coefficient B_1 within a narrow region around zero, the value of ϵ_1 must be limited to be about 0.1 ± 0.2, though in this case, theoretical curves have smaller forward scattering cross sections than the experimental one (see Fig. 4c).

$d\sigma/d\Omega$ at large angles can be diminished by the use of somewhat larger value for $-{}^3\delta_1{}^3$ and ${}^1\delta_1{}^1$. These circumstances are exemplified in Fig. 5–a.

It seems impossible to increase theoretical value of $d\sigma/d\Omega$ at small angles to that of experimental values by some change of $T=0$ state scattering parameters. Therefore we are obliged to change values of $T=1$ parameters to some extent. Such slight changes are done under the restriction that they are consistent with the p-p scattering. B_1 and B_2 are sensitive to the mixing parameter ϵ_2, and larger values of ϵ_2 give smaller backward and larger

forward *n-p* scattering. Changes of the ^3P-phase shifts are also effective to shift the angular distribution curve as shown in Fig. 5–b.

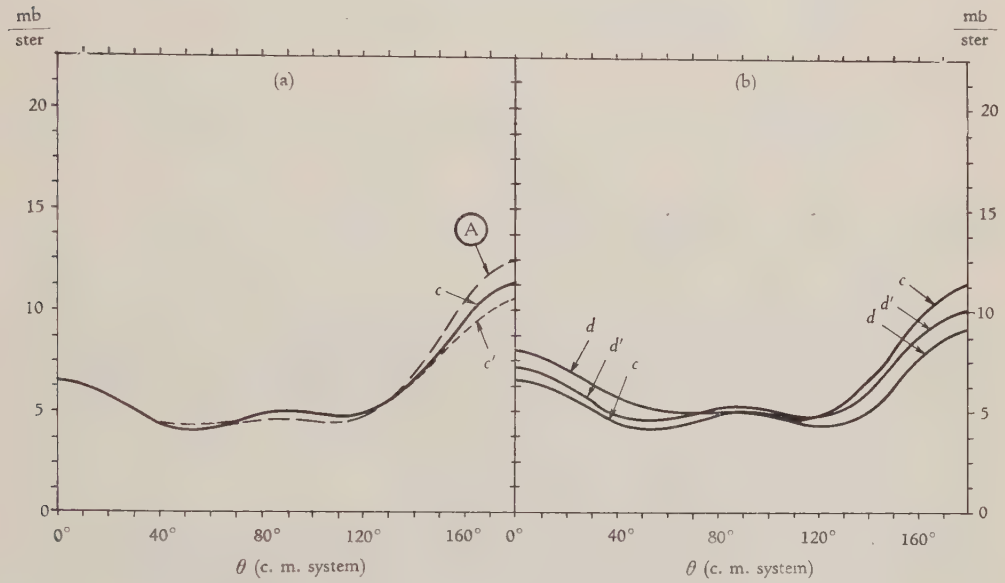


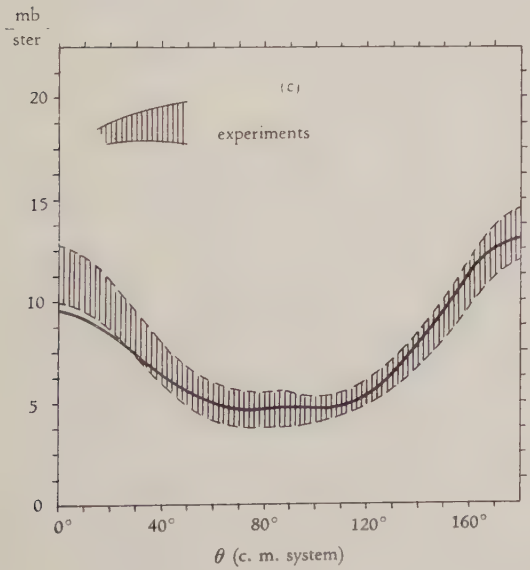
Fig. 5 *n-p* angular distributions.

a) Effects of changes in $T=0$ parameters. Curve Ⓐ is that of Table III with $\epsilon_1=0.09$. Curves *c* and *c'* are calculated using $T=1$ phase shifts of Table II-*c*, and

$^3\delta_1{}^\alpha=0.60$ $^3\delta_1{}^\tau=-0.28$ $\epsilon_1=0.09$ and $^1\delta^1{}_1=-0.25$ for *c*

$^1\delta^1{}_1=-0.20$ for *c'*

b) Effects of changes in $T=1$ parameters on *n-p* angular distributions. Curves are plotted for



$^3\delta_1{}^\alpha=0.62$ $^3\delta_1{}^\tau=-0.25$ and $\epsilon_1=0.18$ and $T=1$ parameters of Table II.

c) The curve is calculated using phase shifts given in page 9.

From these results, we may estimate the final values of the phase shifts, which reproduce all nucleon-nucleon scattering data at 90 MeV and consistent with the pion-theoretical potential. They are as follows :

	$^3F_4$	$^3F_3$	$^3D_3$	$^3D_2$	$^3P_2+^3F_2$	$^3P_1$	$^3P_0$	$^3S_1+^3D_1$
state								
phase shift	0.00	−0.03	0.00	0.29	$^3\delta_2^{\alpha}=0.08$	−0.26	0.52	$^3\delta_1^{\alpha}=0.74$
					$^3\delta_2^{\gamma}=-0.04$			$^3\delta_1^{\gamma}=-0.28$
					$\epsilon_2=-0.18$			$\epsilon_1=-0.07$

state	$^1G_4$	$^1F_3$	$^1D_2$	$^1P_1$	$^1S_0$
phase shift	0.00	−0.04	0.05	−0.17	0.20.

Theoretical curves calculated with these values are plotted in Fig. 1 (polarization) and in Fig. 5-c (angular distribution) and are compared with the experimental data. It is to be noted that the quantitative conditions imposed by the 95 MeV polarization data in section 2 are too strict for scattering parameters at 90 MeV, since $P(\theta)$ increases rapidly with the energy. If the conditions for the polarization are not taken seriously the fit of the angular distribution may be improved.

Of course, these values of phase shifts have some uncertainty. The phase shifts of high angular momentum waves have some uncertainty mainly due to the uncertainty of the coupling constant. These uncertainties may have some effects on the results through the interference terms with $L=1$ and 0 waves. The lower wave phase shifts also have the larger uncertainty, which comes from the experimental error and the unfixed part of the nuclear forces in the regions II and III. Last values of the triplet odd phase shifts are a little larger than that estimated from the p-p analysis*. At any rate, we see from these results that the values of phase shifts, which are reasonable from the pion-theoretical point of view, can reproduce all the nucleon-nucleon data at and, consequently, up to 90 MeV.

§ 4. Concluding Remarks

We have analysed the n-p scattering data at about 90 MeV, and obtained phase shifts which are not contradictory to the outer part (*i.e.* regions I and II) of the pion-theoretical potential.

In the outer part, especially in the region II ($0.7 < x < 1.5$), the pion-theoretical potential has not precisely been obtained, although its general features are well known. Comparing our results with the TMO potential[4,5], our phase shifts seem to suggest a little weaker fourth order potential (two-pion-exchange potential) than that of the TMO potential. This indication is compatible with the propensity of the FST potential and KMO potential.*

Our phase shifts with $L \geq 1$ are calculated or estimated under the assumption that

* See, for example, Fig. 1 of ref. 6) or Fig. 6 of ref. 7).

the potential inside one half of the impact parameter ($x < b_l/2$) hardly contributes to them. Our results seem to show that this assumption is actually realized. This fact means that the inner potential is not so drastic as to cover the effects of the outer potential. The inner interactions may be tentatively represented by "hard core + square well potentials".

Our small values of ${}^3\delta_1{}^\alpha$ and ${}^1\delta_0{}^0$ indicate the existence of hard-core-like repulsive potentials of radius $x_0 = 0.2 - 0.3$ in the even state potentials.[7),8)] Such repulsive interactions should be regarded as intermediate energy properties of the nucleon core. On the other hand, we have some information on the low energy properites of the inner interactions by analyzing low energy data of the two-nucleon system, especially those of the deuteron. Detailed investigations on the energy dependence of the inner interactions are desirable since it might be connected with the structure of the nucleon core.

This paper was written basing on the discussions with S. Otsuki and R. Tamagaki. The author wishes to acknowledge their co-operation.

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Cluster Sums and Related Coefficients of the Ising Model

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It is the purpose of the present paper to discuss the intuitive meaning and the interrelations of various coefficients of the Ising model. Thus, the intuitive meaning of b_{l_m} and $\bar{b}_{l_m}$, the coefficients of cluster sums, is explained. Van der Waerden-Oguchi's expansion is generalized to the case with magnetic field and the coefficients Ω_{n_m} is defined and its meaning is explained. The equivalence of the calculation of the susceptibility with zero field and the non-crossing chain problem is proved. And a method of evaluation of these coefficients is proposed which requires least labour of counting the graphs by using the decomposition of Fuchs' coefficients γ_k.

The data of our calculations are listed in the Appendix. From these data an outlook of this problem to the future and the theory of the imperfect gases are discussed.

§ 1. Introduction and summary

Ising model was first introduced as a model of ferromagnetics.[1] The problems of alloys and lattice gases were shown to be equivalent to this problem[2] in their mathematical structure. For the problem of square lattice without magnetic field, the partition function was obtained by Onsager,[4] Kaufman,[5] and Kac-Ward,[6] and the spontaneous magnetization by Onsager[7] and Yang.[8] The problem with a finite magnetic field is, however, not yet solved in a closed form.*

To obtain the partition function in the case where the magnetic field exists we must either solve the eigenvalue problem[9] by the perturbation method[10] or count the number of graphs, and express the partition function in double series of the temperature and of the magnetic field. Domb et al.[10-12] deviced a method with use of symmetry by which the labour of counting is decreased and they published many elaborate results concerning the quantitative aspects of thermodynamic quantities by calculating the double series expressing the partition function. The method of expanding the logarithm of the partition function (per spin) in series, however, is better than the method of expanding the partition function itself. The calculation of $\lim_{N\to\infty}(1/N)\log \sum \cdots$ can be performed straightforwardly by picking up the only terms proportional to N (total number of lattice points) after enumerating the graphs on the lattice plane which are defined in respective cases, and some of its coefficients have an intuitive meaning as explained in the text. Further

* Hence we do not know where the magnetization curve has the singularity as the function of the magnetic field, and what kind of the singularity this is. This is important to the problem of imperfect gases.[15-17]

it is easier to calculate thermodynamic quantities from this series.

We may cite various methods of expressing the logarithm of the partition function: the expansion by cluster sum b_l (Fujita and Katsura[13] to b_5), that by irreducible cluster sum ${}_l\beta_k$ (Münster and Sagel[14] to ${}_l\beta_4$), that by Fuchs' coefficient γ_k (Münster and Sagel[14] to γ_4), that by semi-invariant λ_l (Kirkwood[15] to λ_3, Chang[16] to λ_6), etc.

In the present paper the meaning of some of these coefficients is clarified and the interrelations between the coefficients are discussed. In particular, we give the direct and intuitive meanings of $\bar{b}_{lm}$, b_{lm} and Ω_{nm}, which are the coefficients of x^m and f^m of cluster sums and the coefficients of generalized Van der Waerden[21]-Oguchi[22]'s expansion in the case with magnetic field (detailed definitions are given in § 3 and § 4) respectively. Then the equivalence of the calculation of zero-field susceptibility to the problem of non-crossing chains is proved. Moreover, Fuchs' coefficients γ_k are decomposed in two parts and one of these is evaluated exactly using the number of nearest neighbors. With the aid of this decomposition of γ_k, a method to obtain the coefficients with least effort of counting the graphs is proposed. For example, the semi-invariants up to λ_6 can be obtained without counting the number of graphs by this method. This method can be applied also to other lattices, but the application will be postponed to another opportunity. Various coefficients obtained in this way are summarized in the Appendix. The forecast of the possibility of expressing the partition function in a closed form, the theory of condensation, the ring approximation of the irreducible cluster integrals, etc., are discussed in the last two sections.

§ 2. Preliminaries

We consider a two dimensional square lattice whose total number of lattice points is N. The interaction energies between nearest parallel spins and between antiparallel spins are $-J/2$ and $J/2$ respectively. $J>0$ and <0 correspond to ferro- and antiferromagnetic cases respectively. We denote the external magnetic field by H, and magnetic moment of spin by $m(=g\beta)$.

Let

$$x=e^{-2K}=e^{-J/kT} \text{ and } y=e^{-2C}=e^{-2mH/kT}. \tag{2·1}$$

The partition function per spin is given by

$$x^{-1}\,y^{-1/2}\Lambda(x,y)\equiv\lim_{N\to\infty}\left[\mathrm{Tr}\ e^{K\sum' s_i s_j+C\sum s_i}\right]^{1/N}, \tag{2·2}$$

where $s_i(=\pm 1)$ is the spin variable, $\sum'$ $s_i\,s_j$ means the summation over all $(2N)$ nearest bonds, $\sum$ s_i means the summation over all (N) lattice points, Tr means the summation over all (2^N) configurations.

The free energy per spin as a function of temperature and magnetic field is denoted by F and is given by

$$F/kT=-\log\Lambda(x,y)+1/2\cdot\log\ y+\log x. \tag{2·3}$$

The magnetization per spin, M, is given by

$$(1+M)/2=y\,\partial \log \Lambda/\partial y. \tag{2·4}$$

$(1+M)/2$ is denoted also by ρ (specific density in the case of the lattice gas) in this paper. The free energy as a function of temperature and magnetization is denoted by F^* and is given by Legendre transformation.

$$\frac{F^*}{kT}=\frac{F}{kT}+\frac{MH}{kT}=-\log \Lambda(x, M)+\frac{1+M}{2}\log y(x, M)+\log x. \tag{2·5}$$

We have following symmetry relations.

$$\Lambda(x, y)=y\Lambda\left(x, \frac{1}{y}\right), \tag{2·6}$$

$$\Lambda(x, 1)=\frac{(1+x)^2}{2}\Lambda\left(\frac{1-x}{1-x}, 1\right), \tag{2·7}$$

$$\Lambda(x, 1)=x^2\Lambda\left(\frac{1}{x}, 1\right). \tag{2·8}$$

Here we explain the topological terminology used in the present paper (cf. Riddel and Uhlenbeck[19]).

A *graph* is a set of points, lines and connected graphs which are not connected together. A *connected graph* is a set of points and lines connecting some of them, of which any two points can be bridged by appropriate connected lines. An *articulation point* of a graph is a point where the graph could be cut into two or more disjoint graphs by cutting all the lines going to this point. A *star* is a connected graph which has no articulation points. A *closed graph* is a star which has no points to which three or more lines end, i.e., a closed graph is a polygon.

Evidently the next subsumption relation holds:

graph ⊃ connected graph ⊃ star ⊃ closed graph.

In "integral" formulation of cluster sums and of irreducible cluster sums, ((3·4), (5·1), Fig. 4 and Fig. 5) the points in the above definition are referred to *particles*. While in the direct formulation of cluster sums and of generalized van der Waerden-Oguchi's coefficients (in Fig. 1. 2. 3) the points are referred to *positions* on the lattice plane, and hence permitted lines are those which connect nearest lattice positions only.

§ 3. Cluster sum

(2·2) can be rewriten as

$$\log \Lambda(x, y)=(1/N)\log \sum_{\{s\}}^{2^N} \prod_{(i,j)}^{2N} e^{-K(1-s_i s_j)} \prod_{i}^{N} e^{-C(1-s_i)}. \tag{3·1}$$

From (3·1) it can be shown that one $+-$ bond contributes x, one $++$ or $--$ bond contritutes 1, one $-$ spin contributes y, and one $+$ spin contributes 1 *to each* configuration in the summation $\sum_{\{s\}}^{2^N}$. The non-vanishing terms after taking the limit $\lim_{N\to\infty}(1/N)\log \sum_{\{s\}}^{2^N}\cdots$ are the ones which are proportional to N in the summation. We

define the coefficients of expansion $\bar{b}_l$ and b_l by

$$\log \Lambda(x, y) = \sum \bar{b}_l \, y^l$$

$$= \sum b_l \, x^{4l} \, y^l, \qquad (3 \cdot 2)$$

and let

$$\bar{b}_l = \sum_m \bar{b}_{lm} \, x^m. \qquad (3 \cdot 3)$$

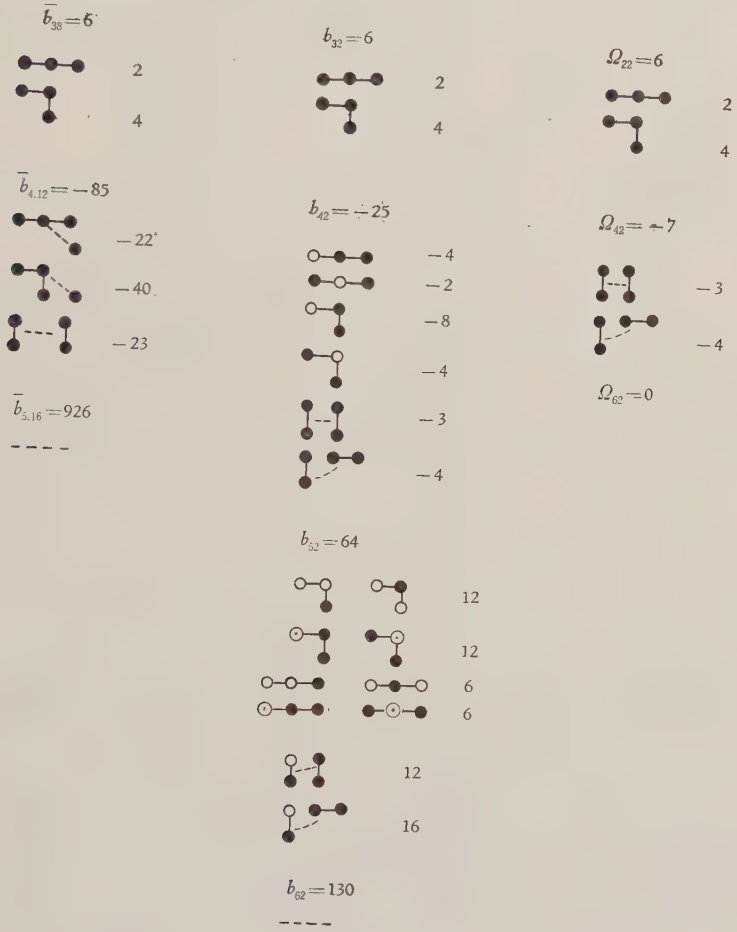


Fig. 1a Fig. 1b Fig. 1c

The graphs to be counted in a) $\bar{b}_{lm}$, b) b_{lm}, and c) Ω_{nm}. The number of bonds is given by $2l-(m/2)$, m and m in a), b) and c) respectively. Fig. 1 shows the case of 2 bonds and Fig. 2 shows the case of 4 bonds. In a) l is the number of points (positions) and in b) l is the number of particles. In a) the disjoint points are contained. In b) and c) disjoint points should not exist. In b) multiple occupancy is permitted and ●, ○ and ⊙ means the singly, doubly and tripply occupied points respectively. In c) n is the sum of the number of end points and that of junction points with three branches. The details concerning signs are explained in the text.

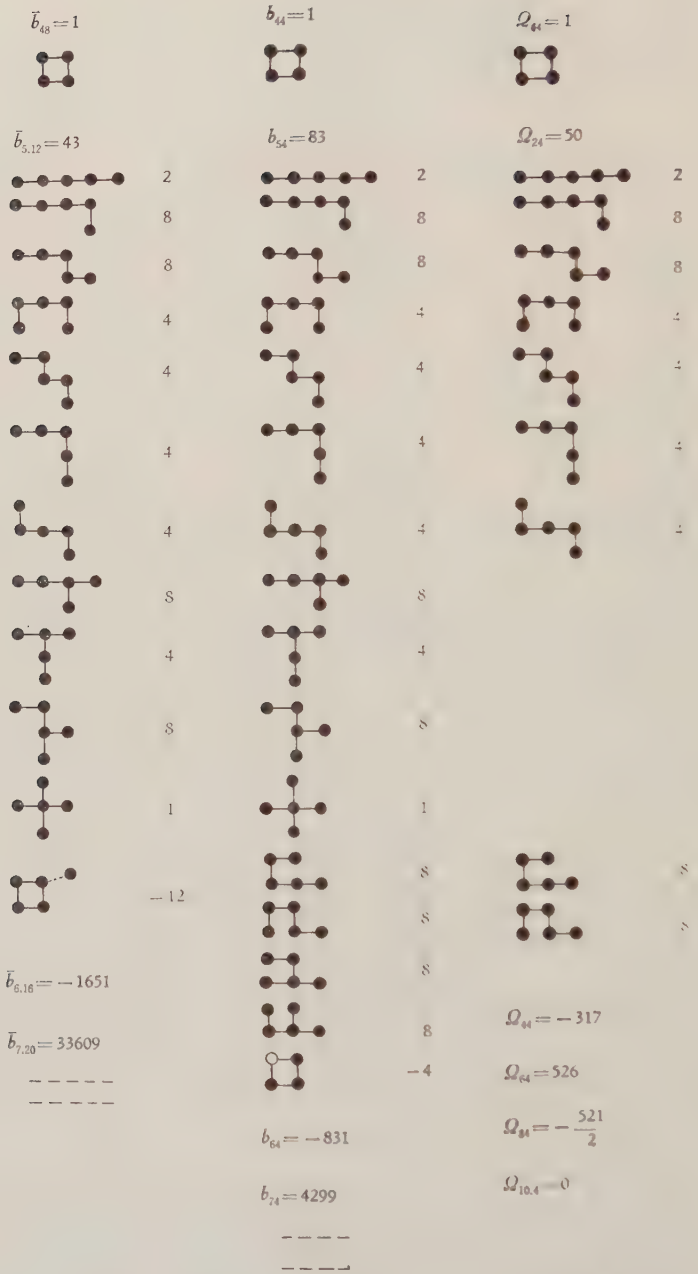


Fig. 2a Fig. 2b Fig. 2c

Thus we see the meaning of $\bar{b}_{lm}$: $\bar{b}_{lm}$ *is the number per lattice point of the graphs each of which consists of* l *points and* $2l-(m\ 2)$ *bonds formed on the lattice plane, and in which*

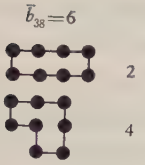


Fig. 3 $\bar{b}_{lm}$
l is the area in the unit of one lattice spacing.
m is the number of bonds.

the nearest lattice points are all connected (Figs 1a and 2a). Alternatively speaking, $\bar{b}_{lm}$ *is a number per lattice point of closed graphs each of which consists of* m *lines, and whose area is* l *in the unit of one lattice spacing*[20] (Fig. 3). Here and throughout in the text *a number per lattice point* means the coefficient of N in the expression as the function of the total number of lattice points N.

On the other hand, we apply the Urcell-Mayer theory of the imperfect gas[17),18)] to the Ising model,[24),25)] b_l is expressed by

$$b_l=\frac{1}{l!\,N}\,\mathsf{S}\sum^{\text{connected}}\!\!\Pi\, f_{ij}, \quad (3\cdot4)$$

where S *means the summation over all manners of arranging different* l *particles in* N *points allowing the multiple occupancy, and* $\sum^{\text{connected}}\!\!\Pi$ *means the summation of the products of* f_{ij}*'s over all connected graphs (clusters) which consist of* l *points* (Fig. 4). This connected graph is irrelevant to the structure of the lattices. Here

$f_{ij}=-1$, i and j particles are at the same point in the graphs,
$=f\equiv x^{-2}-1$, i and j particles are at the nearest points in the graphs,
$=0$, otherwise.

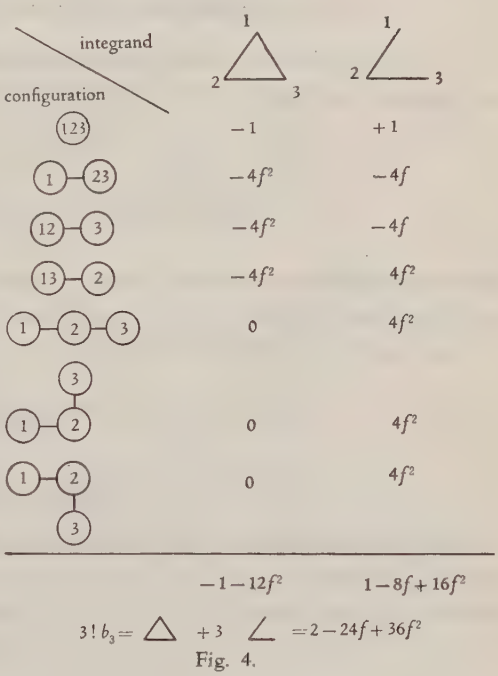


Fig. 4.

When we express $\bar{b}_l$ as a polynomial in x, the term of the highest degree of x corresponds to the graphs which consist of l disjoint points, and hence the highest power of $\bar{b}_l$ is x^{4l}. The term of the lowest degree in x corresponds to the graphs in which the number of bonds connecting l points on the lattice plane become maximum, i.e., $4\sqrt{l}$ or the least even integer exceeding $4\sqrt{l}$. It is denoted by $-2[-2\sqrt{l}\,]$ in Gauss' notation. $[x]$ means the greatest integer not exceeding x. When we express b_l as a polynomial in f, the lowest term is f^0, and the power of the highest term is

$$2l+2[-2\sqrt{l}\,]+1. \quad (3\cdot6)$$

b_{l0} is the cluster sum in the case where $J=0$ (or $T=\infty$) and is given by $-(-)^l/l$.

From the results (cf. Appendix) we have found the direct meaning of the coefficient of b_l in f^m as follows. We shall denote this coefficient by b_{lm}. First we enumerate the graphs consisting of m bonds on the lattice plane such that in this graph disjoint points are excluded and the nearest points need not be connected. The sign $+$ or $-$ is multi-

plied to each graph when the number of connected graphs which construct the concerned graph is odd or even respectively. Next we arrange the same l particles in each of graphs. In this arrangement the multiple occupancy is permitted but the vacant lattice point should not exist. Again the sign $+$ or $-$ is attributed to each lattice point according as the number of particles on that lattice point is odd or even respectively. Then the first and the last signs are multiplied to each number of concerned arrangements. *The algebraic sum of the resulting numbers for given l and m (for $1 \leq m \leq 2l+2\binom{2}{l}-2\binom{N}{l}$) is b_{lm} ($b_{l0}=-(-)^l/l$).* This manner of countings is illustrated in Figs. 1b and 2b.

§ 4. The generalization of van der Waerden-Oguchi's expansion

In this section we consider the generalization of van der Waerden[23]-Oguchi's[22] high temperature expansion to the case where a finite magnetic field exists.† From (2·2),

$$\log \Lambda - \log x - (1/2)\log y$$

$$= \frac{1}{N}\log \sum^{2^N} \prod^{2N} \operatorname{ch} K(1+s_i s_j \operatorname{th} K) \prod^{N} \operatorname{ch} C(1+s_i \operatorname{th} C). \tag{4·1}$$

By performing the product we have

$$= \frac{1}{N}\log\Big[\operatorname{ch}^{2N}K \operatorname{ch}^{N}C$$

$$\times \sum_{s_i=\pm 1}\Big(1+\sum_{\langle i,j\rangle} s_i s_j \operatorname{th} K + \sum'_{\langle ij\rangle}\sum'_{\langle i'j'\rangle}(s_i s_j)(s_{i'} s_{j'})\operatorname{th}^2 K+\cdots\Big)$$

$$\times\Big(1+\sum_i s_i \operatorname{th} C+\sum_i\sum_{i'} s_i s_{i'} \operatorname{th}^2 C+\cdots\Big)\Big], \tag{4·2}$$

where $\sum'_{\langle ij\rangle}$ in the first parenthesis means the summation over all bonds, and $\sum'_i$ in the second parenthesis means the summation over all lattice points. When we develop the product, the coefficient of $\operatorname{th}^m K \operatorname{th}^n C$ is given by

$$\sum (s_i s_j)(s_{i'} s_{j'})\cdots(s_{i(m)} s_{j(m)}) s_k s_{k'}\cdots s_{k(n)}. \tag{4·3}$$

Each term of (4·2) corresponds to a graph which consists of m bonds and n points and in which the nearest lattice points are not neccessarily connected. After performing summation $\sum_{s_i=\pm 1}$ non vanishing terms correspond to the graphs which consist of either the junction point of even bonds or the double point of one end of the bond $(s_i s_j)$ and point s_k.

Thus we see: the coefficient of $\operatorname{th}^m K \operatorname{th}^n C$ in the expansion of $(1/N)\log[\cdots]$, which we denote by Ω_{nm}, *is given by the number per lattice point of graphs which consist of m bonds and in which the sum of the numbers of the end points and the junction points with three (generally odd) branches*†† *is n* (Fig. 1c and 2c). *Here the nearest points need not be connected but the disjoint points should not exist.* Especially $\Omega_{2n+1,m}=0$, and Ω_{0m} *is the number per lattice point of closed graphs consisting of m bonds.*

† Short range order of Cu_3Au was treated by Oguchi.[22]

†† Such a branced graph is seen in for example $\Omega_{43}=-56$.

Hence

$$\log \Lambda - \log x - (1/2)\log y = 2 \log \mathrm{ch}\, K + \log \mathrm{ch}\, C + \log 2 + \sum_{n=0} \sum_{m=0} \Omega_{nm} \mathrm{th}^m K \mathrm{th}^n C. \tag{4·4}$$

The magnetization M and the susceptibility χ per spin are given by

$$M = -\mathrm{th}\, C - \sum_m [2\Omega_{2m} \mathrm{th}\, C + \sum_{n=2} \{-n\Omega_{nm} + (n+2)\,\Omega_{n+2,\,m}\} \mathrm{th}^{n+1} C] \mathrm{th}^m K, \tag{4·5}$$

$$kT\chi = 1 - \mathrm{th}^2 C + \sum_m [2\Omega_{2m} + (-8\Omega_{2m} + 12\Omega_{4m}) \mathrm{th}^2 C + \sum_{n=4} \{(n-2)(n-1)\,\Omega_{n-2,\,m} - 2n^2 \Omega_{nm} + (n+2)(n+1)\,\Omega_{n+2,\,m}\} \mathrm{th}^n C] \mathrm{th}^m K. \tag{4·6}$$

In particular the susceptibility in the case where $H=0$ is given only by Ω_{2m}.

$$kT\chi = 1 + 2\Omega_2 = 1 + \sum_m 2\Omega_{2m} \mathrm{th}^m K. \tag{4·7}$$

Thus it has been proved that the *calculation of the zero-field susceptibility is equivalent to the enumeration of non-crossing chains (two end chains) of given length in the lattice plane.*

Since $\bar{b}_{lm}$ is the number of closed graphs as explained in § 3 it is evident that

$$\sum_{l=1}^{\infty} \bar{b}_{lm} = \Omega_{0m}. \tag{4·8}$$

§ 5. Irreducible cluster sum

The introduction of irreducible cluster sum to lattice system was originated by Fuchs.[24] The irreducible cluster sum β_k is defined by

$$\beta_k(f) = \frac{1}{k!\,N} \mathsf{S} \overset{star}{\sum} \Pi f_{ij}, \tag{5·1}$$

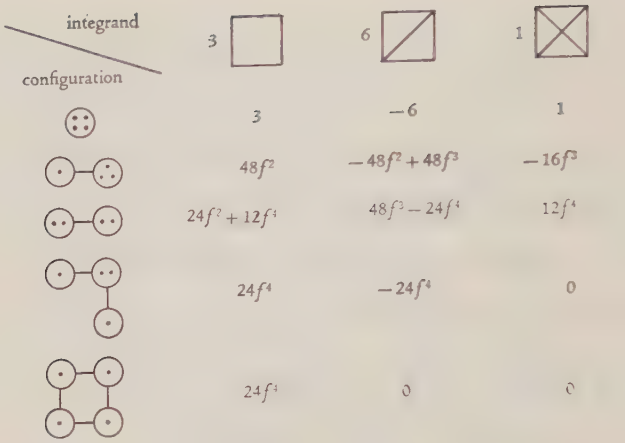
where $\overset{star}{\sum} \Pi$ *means the summation of the products of* f_{ij}*'s over all stars (irreducible clusters) which consist of* $k+1$ *points. The meanings of* f_{ij} *and* S *are as same as those in the case of* b_l. b_l and β_k are connected by[18)19)23)]

$$b_l = \frac{1}{l^2} \sum (\sum k n_k = l-1) \prod_k \frac{(l\beta_k)^{n_k}}{n_k!}. \tag{5·2}$$

The magnetic field as a function of $\rho = \dfrac{1+M}{2}$ with β_k is an antisymmetric function of $M = 1-2\rho$. Therefore $\sum \beta_k \rho^k$ should contain the term $-\log(1-\rho)$, and hence the coefficient of β_k in f^0 is $-1/k$. Defining β_k' and β_{km}' by

$$\beta_k = -\frac{1}{k} + \beta_k', \qquad \beta_k' = \sum \beta'_{km} f^m, \tag{5·3}$$

we have



$$3!\beta_3=-2+24f^2+80f^3+24f^4$$

Fig. 5

$$\log \Lambda(x,\ \rho)=-\log\ (1-\rho)-\sum_{k=1}^{\infty}\frac{k}{k+1}\ \beta_k'\ \rho^{k+1}, \tag{5·4}$$

$$\log \gamma(x,\ \rho)=\log\ \rho-\log(1-\rho)-4\ \log x-\sum_{k=1}^{\infty}\beta_k'\ \rho^k. \tag{5·5}$$

The "integrand" of (5.1) and the configurations which contribute to the integral are shown in Fig. 5.

The result of calculations of ${}_l\beta'_{k,m}$ shows that the lowest power of ${}_l\beta_{k,m}'$ is $-[-k\ 2]$ in Gauss notation and the highest power is

$$m^*(k)=2(k-1)+[-2\sqrt{k-1}]+1. \tag{5·6}$$

First several terms are given by

k	1	2	3	4	5	6	7	8	9	10
$m^*(k)$	1	2	4	5	7	8	10	12	13	15.

These properties of ${}_l\beta_k'$ concerning the highest and the lowest powers in f will be used in the next section to discuss the properties of γ_k.

§ 6. Fuchs' coefficient γ_k and its decomposition

In this section we express the free energy F^* and other quantities in terms of Fuchs' coefficients γ_k and decompose γ_k in two parts γ_k', and γ_k'' and we give the explicit form of γ_k''.

The free energy F^* as a function of magnetization and temperature is

$$\frac{F^*}{kT}=\rho\log\rho+(1-\rho)\log(1-\rho)+8K_I\rho-\sum_{k=1}^{\infty}\frac{\beta_k'\rho^{k+1}}{k+1}+\log x. \qquad (6\cdot1)$$

In general, the term $8K_I\rho$ is replaced by $2zK_I\rho$ where z is the number of nearest neighbors of the lattice. The sum of the third and the fourth terms on the right-hand side of (6·1) should be expressed as a function of $\rho(1-\rho)=(1-M^2)/4$. We denote the $k+1$-th coefficient in the expansion in powers of $(1-M^2)/4$ by γ_k. Then

$$-8K_I\rho+\sum_{k=1}^{\infty}\frac{\beta_k'\rho^{k+1}}{k+1}=\sum_{k=0}^{\infty}\frac{\gamma_k}{k+1}\rho^{k+1}(1-\rho)^{k+1}. \qquad (6\cdot2)^{\dagger}$$

Hence β_k' is expressed in terms of γ_k.

$$\frac{\beta_k'}{k+1}=\sum_{0\leqq\lambda\leqq(k+1)/2}\frac{(-1)^\lambda(k-\lambda)!}{\lambda!\,(k+1-2\lambda)!}\gamma_{k-\lambda}. \qquad (6\cdot3)^{\dagger}$$

Let

$$\rho(1-\rho)=t,\qquad \rho=\frac{1-\sqrt{1-4t}}{2},\qquad \gamma_k=\sum\gamma_{km}f^m. \qquad (6\cdot4)$$

Inserting (6.4) into (6.2), we have

$$\sum_{k=0}^{\infty}\sum_{m}\frac{\gamma_{km}}{k+1}f^m\,t^{k+1}$$

$$=-2\sum_{m=1}^{\infty}\frac{(-1)^m f^m}{m}\left(\frac{1-\sqrt{1-4t}}{2}\right)^m+\sum_{k=0}^{\infty}\sum_{m=-[-k/2]}^{m^*(k)}\frac{\beta_{km}'f^m}{k+1}\left(\frac{1-\sqrt{1-4t}}{2}\right)^{k+1}. \qquad (6\cdot5)$$

The coefficient of $t^{k+1}f^m$ in the expansion on the right-hand side is $\gamma_{km}/k+1$. We denote the part due to the first term of the right-hand side by γ_{km}'', and the remaining part by γ_{km}'.

Then

$$\gamma_{km}''=\frac{k+1}{2\pi i}\oint\frac{(-)^m(1-\sqrt{1-4t})}{t^{k+2}}dt=\frac{2(-)^m}{m}\frac{(2k)!}{(k!)^2}\quad(m=k+1,\ k+2\cdots),$$

$$=0\qquad(m=0,\ 1,\cdots k). \qquad (6\cdot6)$$

γ_{km}' does not vanish only when $m=k+1,\ k+2,\cdots m^*(k)$.
This fact is due to the properties of the highest and the lowest powers of β_k'. We rewrite γ_k as

$$\gamma_k=-8\frac{(2k)!}{(k!)^2}K+2\frac{(2k)!}{(k!)^2}\sum_{m=1}^{k}\frac{(-)^{m-1}f^m}{m}+\sum_{m=k+1}^{m^*(k)}\gamma'_{km}f^m. \qquad (6\cdot7)$$

Hence γ_k is decomposed into γ_k'' and γ_k'.

---

† Fuchs[24] proved (6.3) for $k\geqq3$. (6.3) is, however, valid from β_1 when we consider that γ_k' exists from γ_0 and that (6.2) holds.

The part of F^* due to γ_k'' in the sum in (6.2) can be given in a closed form. We apply the integral representation (6.6) to γ_k'' in the expression $\frac{\partial}{\partial f}\sum\frac{\gamma_k''}{k+1}f^{k+1}\times(1-f)^{k+1}$, change the order of the summation $\sum_k$ and integration, and perform the summation $\sum_{m=1}^{\infty}$ under the integral sign. Then we have by residue theorem

$$\frac{\partial}{\partial f}\sum_{k=0}^{\infty}\frac{\gamma_k''}{k+1}f^{k+1}(1-f)^{k+1}=\frac{1-\sqrt{1+4f(1-f)}}{f(1+f)}$$

$$=\frac{1-M^2}{2(1+f)}[\mathrm{th}\{\tfrac{1}{4}\log(1+f(1-M^2))\}-1]. \tag{6·8}$$

Noting $d/dK=4(1+f)d/df$, then

$$\sum_{k=0}^{\infty}\frac{\gamma_k''}{k+1}f^{k-1}(1-f)^{k+1}=2\int_0^K(1-M^2)\,\mathrm{th}\,\tfrac{1}{2}\log(1-f(1-M^2))-1\,dK \tag{6·9}$$

$$=2\log(\sqrt{1+f(1-M^2)}+1)-2\log 2$$

$$-(1+M)\log(\sqrt{1+f(1-M^2)}+M)+(1+M)\log(1+M)$$

$$-(1-M)\log(\sqrt{1+f(1-M^2)}-M)+(1-M)\log(1-M). \tag{6·10}$$

The γ_k'' contributions to the free energy, the magnetic field, and the susceptibility, expressed as the functions of magnetization, are summarized in closed forms and the whole expressions are given as follows.

$$\frac{F^*}{kT}=\frac{1+M}{2}\log\frac{1+M}{2}+\frac{1-M}{2}\log\frac{1-M}{2}$$

$$-2\int_0^K(1-M^2)[\mathrm{th}\,\tfrac{1}{4}\log(1+f(1-M^2))]dK-\sum_{k=0}^{\infty}\frac{\gamma_k'}{k+1}\left(\frac{1-M^2}{4}\right)^{k+1}, \tag{6·11}$$

$$\frac{H}{kT}=-\frac{1}{2}\log y$$

$$=-\log\frac{\sqrt{1+f(1-M^2)}+M}{\sqrt{1+f(1-M^2)}-M}+\frac{1}{2}\log\frac{1+M}{1-M}-\frac{M}{2}\sum_{k=0}^{\infty}\gamma_k'\left(\frac{1-M^2}{4}\right)^k, \tag{6·12}$$

$$\frac{1}{kT\chi}=\frac{1}{(1-M^2)\sqrt{1+f(1-M^2)}}-\frac{1}{1-M^2}$$

$$+\frac{1}{2}\sum_{k=0}^{\infty}\gamma_k'\left(\frac{1-M^2}{4}\right)^k-\frac{M^2}{4}\sum_{k=0}^{\infty}k\gamma_k'\left(\frac{1-M^2}{4}\right)^{k-1}. \tag{6·13}$$

When we neglect γ_k' contribution, the Curie point is given by $x_c=0.5$, which is as same as the Bethe-Fowler-Takagi approximation.

§ 7. Coefficients Γ'_{jm} and Γ_{jm}

In this section we rearrange the series $\sum_k \sum_m \frac{\gamma'_{km}}{k+1}\left(\frac{1-M^2}{4}\right)^{k+1} f^m$ and define the new coefficient Γ_{jm} and relate it to the number of graphs in the lattice plane.

Since the highest and the lowest powers of β_{km}' are $-[-k/2]$ in Gauss notation and $m^*(k)$ respectively, the highest and the lowest powers of γ_{km}' are $k+1$ and $m^*(k)$ respectively. When we rearrange the double series

$$\sum_k \sum_m \frac{\gamma'_{km}}{k+1}\left(\frac{1-M^2}{4}\right)^{k+1} f^m \tag{7·1}$$

into

$$\sum_{j=0}^{\infty}\left(\frac{4}{1-M^2}\right)^j \sum_{k=m^*(k)}^{\infty} \frac{\gamma_{k,k+j+1}}{k+1}\left(\frac{f(1-M^2)}{4}\right)^{k+j+1}, \tag{7·2}$$

then the series in the second summation symbol becomes a function of only one variable $\frac{f(1-M^2)}{4}$. We expand this series in porwer series of

$$u \equiv \text{th}\left[\tfrac{1}{4}\log\{1+f(1-M^2)\}\right]$$

and denote the coefficient by Γ''_m. Further we expand this result into the double series of $\frac{4M^2}{1-M^2}$ and u, and denote its coefficient by Γ_{jm}. Then

$$(7·1) = (7·2) = \sum_{j=0}^{\infty}\left(\frac{4}{1-M^2}\right)^j \sum_m \Gamma_{jm}'\, u^m \tag{7·3}$$

$$= \sum_{j=0}^{\infty}\left(\frac{4M^2}{1-M^2}\right)^j \sum_m \Gamma_{jm}\, u^m. \tag{7·4}$$

When we put $M=0$ in (7.4) and insert it into (6.11), F^*/kT should reduce to the Onsager Kac-Ward solution, and hence Γ'_{0m} is identified to the number of closed graphs ($=\Omega_{0m}$ defined in § 4).

We could not find the intuitive meaning of Γ'_{jm} or Γ_{jm} for $j \geqq 1$ in general. We find, however, that the first non-vanishing Γ'_{jm} for given j is the number of graphs (in this case *stars*) which consist of m bonds and of least number of points.

Free energy, magnetic field, and susceptibility can be expressed in terms of Γ'_{jm}, and the part due to Γ'_{0m} can be expressed in a closed form. For example the susceptibility without magnetic field is,

$$\frac{1}{kT}\frac{1}{\chi} = 2\frac{1-\text{th}\,K}{1+\text{th}\,K} - 1 + \frac{1-3\,\text{th}^2\,K}{(1+\text{th}\,K)^2} + \frac{(1+\text{th}^2\,K)(2\,\text{th}^2\,2K-1)}{(1+\text{th}\,K)^2}\,\frac{2}{\pi}\,\mathbf{K}(k(K)) - 8\sum_{m=7}^{\infty}\Gamma_{1m}\,\text{th}^m K, \tag{7·5}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind, and $k=2\,\mathrm{sh}\,2K\,\mathrm{ch}^2\,2K$.

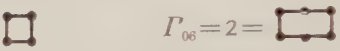
§ 8. Evaluation of the various coefficients

The evaluations of the expansion coefficients b_l, β_k, Ω_n, γ_k, Γ_j, etc., are equivalent to each other, i.e., a knowledge of any coefficient to m-th power of respective parameter (x, f, K, th K; C, y, th C, etc.) enables us to calculate all other coefficients to m-th power of respective parameter.

Now we propose a method of the evaluation with least labour of the enumeration of the graphs.

First we consider the expansion by Γ_{jm}. From the meaning of Γ_{0m} (the number of closed graphs) or from Onsager's solution we see

$$\Gamma_{04}=1=\quad \Gamma_{06}=2= \tag{8·1}$$



and

$$\Gamma_{0m}=0 \qquad \text{for} \qquad m=1,\,2,\,3,\,5.$$

And non-vanishing Γ_{1j} begins from $m=7$.

Thus from (7.4),

$$\gamma_{34}'=\beta_{34}'=4,\quad \gamma_{45}'=\beta_{45}'=-40,\quad \gamma_{56}'=\beta_{56}'=276. \tag{8·2}$$

Using γ_k' given in (8.2) and γ_k'' in (6.6), all β_{km}' and b_{lm} up to $m=6$ can be obtained from (6.3) and (5.2). The semi-invariant λ_i can be given by the relation

$$\sum_{k=0}^{\infty}\frac{\gamma_k}{k+1}\rho^{k+1}(1-\rho)^{k+1}=\sum_{i=1}^{\infty}\frac{\lambda_i}{i!}(-4K)^i. \tag{8·3}$$

Thus β_k, b_l, λ_i, can be evaluated up to β_4, b_7, λ_{10}, respectively, only from data (8.1). If we regard the Onsager's solution as known, the above coefficients can be given without counting any graphs.

Further, we know (directly or from Onsager's solution)

$$\Gamma_{07}=0,\qquad \Gamma_{08}=9/2,\qquad \Gamma_{09}=0. \tag{8·4}$$

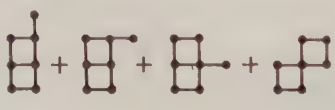
The first Γ_{2m} which does not vanish is $\Gamma_{2\cdot 10}$.

Thus the data of Γ_{17}, Γ_{18}, and Γ_{19} enables one to calculate all coefficients up to the 9th power of the temperature parameter (f^9, K^9, th ^9K). The necessary b_{lm}'s are

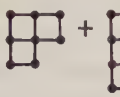
$$b_{6,\,10}=2=$$

,

$$b_{7,8}=22=$$



,

$\bar{b}_{8,12}=6=$  ,

$\bar{b}_{8,14}=134$ (the number of graphs which consist of 8 points and 9 bonds in the lattice plane). (8·5)

The direct counting of graphs corresponding to these $\bar{b}_{lm}$ enables us to calculate Γ_{17}, Γ_{18}, Γ_{19}, and all expansion coefficients γ'_{km}, β'_{km}, b_{lm}, Ω_{nm} up to $m=9$. Thus b_l, β_k', λ_i, are completely determined up to b_8, β_7, λ_9 respectively. In the Appendix the values of b_l, β_k', γ_k', Ω_n, Γ_j', Γ_j and λ_i, are listed. The values up to b_5 agree with the values obtained by the direct counting of graphs by Fujita and the present author-[13] Further we transform Domb's coefficients (summarized in Appendix 1 of reference 11) into b_l and compare it with ours and ensured the coinsidence up to $l=8$. β_k and γ_k are obtained by Münster and Sagel[14] up to $k=4$, and we find their β_4 and γ_4 should be corrected. The semi-invariant λ_i are obtained by Chang†[16] up to λ_6, which agree to ours up to λ_6.

In conclusion we can say: In order to evaluate the expansion coefficients up to m'th power of temperature variable, (f, K, th K etc), it is sufficient to know Γ_{jm} up to $m=m'$. When we regard Γ_{0m} as known it is sufficient to know β'_{lm} or b_{lm} for $m' \geqq m \geqq l+1$ and hence to know the first $l-m'$ (>0) terms of $\bar{b}_{lm}$ by direct counting of graphs. This labour of enumeration is much less than the one which Domb's method[10] requires in the evalution of the coefficient of $\Lambda(x, y)$.

Here we have used the symmetry of F^* with respect to ρ and $1-\rho$, the number of nearest neighbers, and Onsager's solution in the form of high temperature expansion. Besides we have the knowledge of Onsager's solution in low temperature expansion and Yang's spontaneons magnetization. When we use the coefficients of these expansions, the number of graphs to be enumerated can be decreased further. However we did not use these expansions since a systematic method could not be found.

§ 9. Discussions

Well-known beautiful results of the Ising model of the free energy and the magnetization have been obtained in the case where the magnetic field does not exist. In fact, it might be a fortunate accident that the partition function without magnetic field is reduced to a combination of elementary functions, and we might not expect such circumstances in the case with finite magnetic field.

We notice, however, all existing theories (Kramers-Wannier, Onsager, Yang) follow the method in which first the partition function of finite system is obtained and at the last stage the limit of infinite volume is taken. It is desirable that at a very early stage of the formulation the limit of infinite volume is taken and the functional equation for

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† In Chang's general results[16], $\theta=\theta'$, $z=4$, $y=8$, $\gamma_1=0$, $\gamma_2=24$.

the partition function per spin is established. This functional equation may be an integral equation or a difference equation. At that time, the coefficients of the double series expressing the partition funtion may be obtained by a general formula or at least by a recurrence formula, even if the solution of this functional equation is not reduced to a known function. Then we may regard the problem as solved for the first time.

The method developed in this paper would be the first attempt toward this end. There is obtained an expansion of which the coefficients, such as $\bar{b}_{lm}$, b_{lm} and Ω_{nm} have intuitive meanings as described in the preceding sections. In Figs. 1a, 1b, 1c and 2a, 2b, 2c we compare these different manners of counting graphs in the case of 2 bonds and 4 bonds respectively.

Unfortunately we could not find the direct meanings of ${}_i\bar{\gamma}'_{jm}$, ${}_i\gamma'_{km}$, Γ'_{nm}, and Γ'_{jm} except Γ'_{0m} and Γ'_{jm} which does not vanish for the first time for given j. It is desirable that some expansions will be found in which the coefficient is expressed only by the number of *stars* in the lattice plane. For example, the decomposition of Ω_{nm} may be hopeful. A little promising method to obtain the solution in a closed form is to express the partition function in a determinant form whose expansion coefficients are either: 1) the number of closed graphs b_{lm}, as a function of the number of sides $2l-(m-2)$ and area l, or 2) the number of graphs Ω_{nm} which consists of m bonds and whose sum of the number of end points and the number of branch points with three branches are n, or 3) Γ_{jm} for $j \geqq 1$ explained in § 7, etc. The determinant which is formed of any of these will be the generalization of the Kac-Ward determinant. The attack along this line is expected in future.*

§ 10. Note on the theory of imperfect gases

As described at the footnote in the introduction, we do not know whether the condensation point is a singularity or not in the problem of imperfect gases. It should be noticed that the Yang-Lee theory of condensation does not answer to this problem.[28] Here we remark a few points from the results of our calculations.

In the distribution of the signs of the coefficients of the data of b_{lm} and $\bar{b}_{lm}$ regularity is hard to be found. In ${}_i\bar{\gamma}'_{km}$ and ${}_i\gamma'_{km}$, the terms for odd k are negative and those for even k are positive up to ${}_i\bar{\gamma}'_{k0}$, ${}_i\gamma'_{k0}$, and up to ${}_i\bar{\gamma}'_{2m}$, ${}_i\gamma'_{2m}$. These properties, however, are destroyed at ${}_i\bar{\gamma}'_{8,12}=+9$, ${}_i\bar{\gamma}'_{9,13}=-60$. The signs of the coefficients of Ω_{lm} and λ_{lk} are distributed regularly in our results.

The b_l as a function of temperature is shown in Fig. 5. b_l (up to b_8) is positive below the Curie point, and is an oscillating function above the Curie point. In the Husimi-Temperley model, this property has been proved.[27] And it is surmised that zero of $b_l(T)$ has a lower bound also from Kihara-Kaneko's results[29] for the cluster integral of the gases with square well potential. It seems from the curve of $b_l(T)$ in Fig. 6

*Temperley has informed the author that he has attacked along such line (private communication).

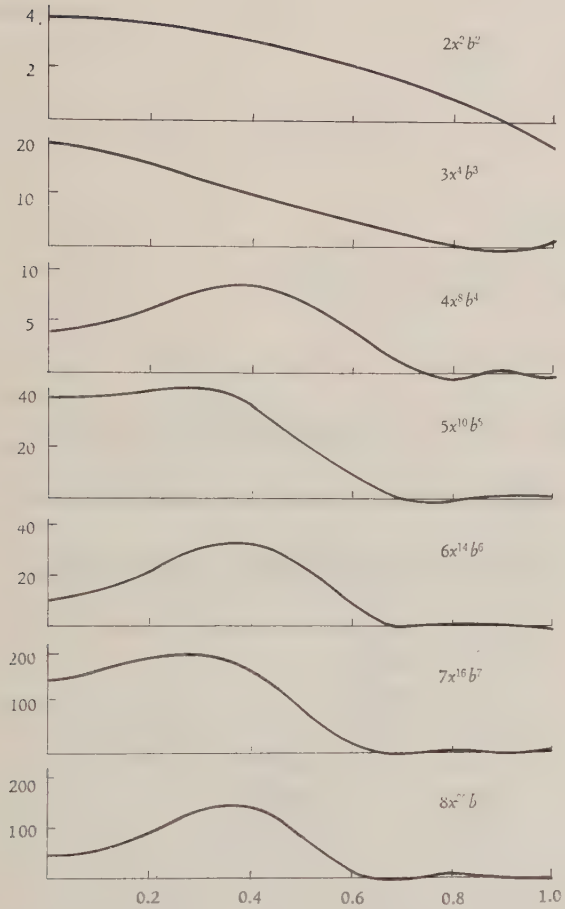

Fig. 6. Temperature dependence of the cluster sum. The abscisa denotes $x=e^{-2K}=e^{-J/kT}$. $b^2 \sim b^8$ should be read as $b_2 \sim b_8$.

that the higher b_l's do not contribute above the Curie point.

On the other hand, β_k has no such a property, that is, there exists such β_k's which are negative even at sufficiently low temperatures. Thus the usually accepted proposition, "cluster integral is always positive for sufficiently low temperature", is not valid for β_k. Hence the relation between the condensation point (denoted by ρ_c) and the singularity of $\sum \beta_k \rho^k$, which has the smallest absolute value (denoted by ρ_s), has the next six possibilities, and we cannot select the one in the present stage: 1) The singularity does not exist for $|\rho|<1$ (the case of Husimi-Temperley model[27]). 2) The singularity is not real and exists outside the circle of $R=\rho_c$ (the case of ideal Bose gas)[28] 3) The singularity exists on the circle of $R=\rho_c$ but is not real positive. 4) ρ_c is a singularity, but there exists another singularity on the circle of $R=\rho_c$. 5) ρ_c is a singularity and there exists no singularity on the circle $R=\rho_c$ except ρ_c. 6) The singularity is not real and exists inside the circle of $R=\rho_c$.

Ring approximation for ${}_l\beta_k$: In Mayer's theories of imperfect gases, the evaluation of the higher terms of ${}_l\beta_k$ is generally difficult, not only in the integration but also in writing down the integrand. Therefore the approximation of ${}_l\beta_k$ to the ring part of the integrand due to Montroll-Mayer[30] is often used. When we compare the exact ${}_l\beta_3$ and the ring approximation of β_3,

$$\beta_3 = -\frac{1}{3}+4f^2+\frac{40}{3}f^3+4f^4,$$

$$\beta_3^{Ring}=\frac{1}{2}+12f^2 \qquad +10f^4.$$

The approximation is not good. Thus the solution of Green's linearized integral equation[31] for the imperfect gas, which is equivalent to the sum of the contribution of β_k in the ring approximation,[32] would give a fairly different isotherm from the real one.

Appendix

Cluster sum $\bar{b}_l$ in powers of x. $\bar{b}_l=\sum \bar{b}_{lm}\, x^m$.

$$\bar{b}_1=x_4$$

$$\bar{b}_2 = 2x^6-\frac{5}{2}x^8$$

$$\bar{b}_3 = \quad 6x^8-16x^{10}+\frac{31}{3}x^{12}$$

$$\bar{b}_4 = \quad x^8+18x^{10}-85x^{12}+118x^{14}-\frac{209}{4}x^{16}$$

$$\bar{b}_5 = \quad 8x^{10}+43x^{12}-400x^{14}+926x^{16}-872x^{18}+\frac{1476}{5}x^{20}$$

$$\bar{b}_6 = \quad 2x^{10}+40x^{12}+30x^{14}-1651x^{16}+\frac{17978}{3}x^{18}-9144x^{20}$$

$$\bar{b}_7 = \quad 22x^{12}-136x^{14}-486x^{16}-5644x^{18}+33609x^{20}$$

$$\bar{b}_8 = \quad 6x^{12}-134x^{14}-389x^{16}-3986x^{18}-13323x^{20}$$

$$\bar{b}_9 = \quad x^{12}+72x^{14}+\cdots\cdots$$

$$\bar{b}_{10} = \quad 30x^{14}-\cdots$$

$$\bar{b}_{11} = \quad 8x^{14}+\cdots\cdots$$

$$\bar{b}_{12} = \quad 2x^{14}+\cdots\cdots$$

$$\bar{b}_{13} = \quad 68x^{16}-\cdots$$

$$\bar{b}_{14} = \quad 22x^{16}+\cdots\cdots$$

$$\bar{b}_{15} = \quad 6x^{16}-\cdots$$

$$\bar{b}_{16} = \quad x^{16}+\cdots\cdots$$

$$\bar{b}_{17} = \quad 88x^{18}+\cdots\cdots$$

$$\bar{b}_{18} = \quad 30x^{18}-\cdots$$

$$\bar{b}_{19} = \quad 8x^{18}+\cdots\cdots$$

$$\bar{b}_{20} = \quad 2x^{18}+\cdots\cdots$$

$$\sum_{l=1}\sum_{m=1}\bar{b}_{lm}\, x^m = x^4+2x^6+\frac{9}{2}x^8+12x^{10}+\frac{112}{3}x^{12}+130x^{14}+\cdots\cdots$$

$$\sum_{l=1}\sum_{m=1} l\,\bar{b}_{lm}\, x^m = 2x^4+8x^6+34x^8+152x^{10}+714x^{12}+3472x^{14}+\cdots\cdots$$

Cluster sum b_l in powers of f, $b_l=\sum b_{lm} f^m$, $f=x^{-2}-1$, $(\bar{b}_l=b_l\, x^{4l})$.

$$b_1=1$$

$$b_2=-1/2\ +2f$$

$$b_3=\ 1/3\ -4f\ +\ 6f^2$$

$$b_4=-1/4\ +6f\ -\ 25f^2+\ 22f^3+\ f^4$$

$$b_5=\ 1/5\ -8f\ +\ 64f^2-\ 148f^3+\ 83f^4+\ 8f^5$$

$$+ \quad 6520x^{22} \quad - \quad \frac{10739}{6}x^{24}$$

$$- \ 75640x^{22} \ + \quad 85954x^{24} - 49328x^{26} + \ \frac{79780}{7}x^{28}$$

$$+ 164790x^{22} \ - \frac{1064393}{2}x^{24} + 87670x^{26} - 785091x^{28} + 377040x^{30} - \frac{601905}{8}x^{32}$$

(A·1)

(A·2)

(A·3)

$$b_6 = -1/6 + 10f - 130f^2 + \frac{1676}{3}f^3 - 831f^4 + 312f^5 + 54f^6$$
$$b_7 = 1/7 - 12f + 230f^2 - 1572f^3 + 4299f^4 - 4492f^5 + 1082f^6$$
$$b_8 = -1/8 + 14f - 371f^2 + 3682f^3 - \frac{31685}{2}f^4 + 30434f^5 - 23263f^6$$
$$b_9 = 1/9 - \cdots$$
$$b_{10} = -1/10 + \cdots$$
$$b_{11} = 1/11 - \cdots$$
$$b_{12} = -1/12 + \cdots$$

Coefficients $Q_n = \sum Q_{nm} \operatorname{th}^m K$.

$$Q_0 = \operatorname{th}^4 K + 2\operatorname{th}^6 K$$
$$Q_2 = 2\operatorname{th} K + 6\operatorname{th}^2 K + 18\operatorname{th}^3 K + 50\operatorname{th}^4 K + 138\operatorname{th}^5 K + 370\operatorname{th}^6 K$$
$$Q_4 = -7\operatorname{th}^2 K - 56\operatorname{th}^3 K - 317\operatorname{th}^4 K - 1512\operatorname{th}^5 K - 6493\operatorname{th}^6 K$$
$$Q_6 = \frac{116}{3}\operatorname{th}^3 K + 526\operatorname{th}^4 K + 4484\operatorname{th}^5 K + 30248\operatorname{th}^6 K$$
$$Q_8 = -\frac{521}{2}\operatorname{th}^4 K - 5072\operatorname{th}^5 K - 58238\operatorname{th}^6 K$$
$$Q_{10} = \frac{9812}{5}\operatorname{th}^5 K + 49992\operatorname{th}^6 K$$
$$Q_{12} = -\frac{47644}{3}\operatorname{th}^6 K$$

$$\sum_{n=0}^{\infty} Q_{nm} = (-)^{m-1}\frac{2}{m}.$$

Irreducible cluster sum $\beta_k' = \beta_k + \frac{1}{k} = \sum \beta'_{km} f^m$.

$$\frac{\beta_1'}{2} = 2f$$
$$\frac{\beta_2'}{3} = -2f^2$$
$$\frac{\beta_3'}{4} = f^2 + \frac{10}{3}f^3 + f^4$$
$$\frac{\beta_4'}{5} = -4f^3 - 11f^4 - 8f^5$$
$$\frac{\beta_5'}{6} = \frac{4}{3}f^3 + 20f^4 + \frac{284}{5}f^5 + 46f^6 + 2f^7$$

$$+ \quad 2f^7$$

$$+ \; 312f^7 + \quad 22f^8$$

$$+3114f^7+\frac{3341}{2}f^8+ \; 194f^9+6f^{10}$$

$$\cdots \quad +84f^{11}+f^{12}$$

$$\cdots +30^{13}$$

$$\cdots +8f^{15}$$

$$\cdots +2f^{17} \qquad (A\cdot 4)$$

$$+ \qquad +\frac{9}{2}\mathrm{th}^8 K \qquad +12\mathrm{th}^{10}K \quad +\frac{112}{3}\mathrm{th}^{12}K \quad +130\mathrm{th}^{14}K+\cdots$$

$$+986\mathrm{th}^7 K +2586\mathrm{th}^8 K+6746\mathrm{th}^9 K+\cdots$$

$$-\cdots\cdots$$

$$+\cdots\cdots$$

$$-\cdots\cdots$$

$$+\cdots\cdots$$

$$-\cdots\cdots \qquad (A\cdot 5)$$

$$(A\cdot 6)$$

$$\frac{\beta_6'}{7} = -14f^4-128f^5-320f^6-252f^7-34f^8$$
$$\frac{\beta_7'}{8} = \frac{7}{2}f^4+134f^5+855f^6+\frac{12828}{7}f^7+\frac{2937}{2}f^8$$
$$\frac{\beta_8'}{9} = -68f^5-\frac{3530}{3}f^6-5652f^7-\frac{21831}{2}f^8$$
$$\frac{\beta_9'}{10} = \frac{68}{5}f^5+\frac{2686}{3}f^6+9574f^7+37646f^8$$
$$\frac{\beta_{10}'}{11} = -360f^6-9660f^7-75012f^8$$
$$\frac{\beta_{11}'}{12} = 60f^6+5822f^7+93674f^8$$
$$\frac{\beta_{12}'}{13} = -1944f^7-74921f^8$$
$$\frac{\beta_{13}'}{14} = \frac{1944}{7}f^7+37458f^8$$
$$\frac{\beta_{14}'}{15} = -10702f^8$$
$$\frac{\beta_{15}'}{16} = \frac{5351}{4}f^8$$
$$\frac{\beta_{16}'}{17} =$$
$$\frac{\beta_{17}'}{18} =$$

$$\sum_{k=1}^{\infty}\beta'_{km}=-(-)^m\frac{4}{m},\quad \sum_{k=1}^{\infty}\beta'_{km}\left(\frac{1}{2}\right)^k=-(-)^m\frac{2}{m},\quad \sum_{k=1}^{\infty}\frac{\beta'_{km}}{k+1}$$
$$\sum_{k=1}^{\infty}k\beta'_{km}=0\quad (m=2.3.4\cdots)$$

Semi-invariant λ_i, $t=v(1-v)=\frac{1-M^2}{4}$. λ_i, N in Kirkwood-Chang's paper is

$$\lambda_1=2t$$
$$\lambda_2=2t^2$$
$$\lambda_3=-2t^2+8t^3$$
$$\lambda_4=2t^2-24t^3+84t^4$$
$$\lambda_5=-2t^2+56t^3-600t^4+1632t^5$$
$$\lambda_6=2t^2-120t^3+3060t^4-21120t^5+43200t^6$$
$$\lambda_7=-2t^2+248t^3-13800t^4+178080t^5-856800t^6+1399680t^7$$
$$\lambda_8=2t^2-504t^3+58884t^4-1243200t^5+10503360t^6-39231360t^7$$
$$-\log x+\sum_{i=1}^{\infty}\sum_{k=2}^{\infty}\lambda_{ik}\frac{(-4K)^i}{i!}\left(\frac{1}{4}\right)^k=K^2+\frac{5}{6}K^4+\frac{32}{45}K^6+\frac{425}{252}K^8+\cdots\cdots$$

$$+ \quad 354f^9 + \quad 6f^{10}$$

$$- \quad 9216f^9 - \cdots\cdots \quad + f^{12}$$

$$+ \quad \frac{616036}{9} f^9 + \cdots\cdots \quad - 6f^{13}$$

$$- \quad 256720f^9 - \cdots\cdots \quad + 8f^{15}$$

$$+ \frac{1744988}{3} f^9 + \cdots\cdots \quad + 2f^{17}$$

$$- \frac{2577344}{3} f^9 - \cdots\cdots$$

$$+ \quad 851696f^9 + \cdots\cdots$$

$$- \frac{1695056}{3} f^9 - \cdots\cdots$$

$$+ \quad \frac{724568}{3} f^9 + \cdots\cdots$$

$$- \quad 60316f^9 - \cdots\cdots$$

$$\frac{60316}{9} f^9 + \cdots\cdots \qquad \text{(A·7)}$$

$$= -(-)^m \frac{2}{m}, \qquad (m = 1, 2, 3\cdots) \qquad \text{(A·8)}$$

$$\text{(A·9)}$$

denoted by λ_t in this paper. $\lambda_t = \sum \lambda_{tk} t^k$.

$$+ 53938080t^8 \qquad \text{(A·10)}$$

$$\text{(A·11)}$$

Coefficients $\gamma_k' = \sum \gamma'_{km} f^m$, part of Fuchs' coefficients.

$$\frac{\gamma_3'}{4} = f^4$$

$$\frac{\gamma_4'}{5} = -8f^5$$

$$\frac{\gamma_5'}{6} = 46f^6 + 2f^7$$

$$\frac{\gamma_6'}{7} = -240f^7 - 34f^8$$

$$\frac{\gamma_7'}{8} = \frac{2461}{2}f^8 + 354f^9 + 6f^{10}$$

$$\frac{\gamma_8'}{9} = -6384f^9 + \textit{term of order } f^{10} - \frac{39}{2}f^{11}$$

$$\frac{\gamma_9'}{10} = \textit{term of order } f^{10} \cdots\cdots$$

Coefficients $\Gamma_j' = \sum \Gamma_{jm}' u^m$ and $\Gamma_j = \sum \Gamma_{jm} u^m$, where $u = \text{th}\left[(1/4)\log\{1+f\right.$

$$\Gamma_0' = u^4 + 2u^6 - 8u^7 + \frac{57}{2}u^8 - 80u^9 \cdots\cdots$$

$$\Gamma_1' = 2u^7 - 6u^8 + 20u^9 \cdots\cdots$$

$$\Gamma_2' = 6u^{10} + \frac{201}{2}u^{11} \cdots\cdots$$

$$\Gamma_3' = u^{12}$$

$$\Gamma_0 = u^4 + 2u^6 + \frac{9}{2}u^8 + 12u^{10} - \frac{112}{3}u^{12}$$

$$\Gamma_1 = 2u^7 - 6u^8 + 20u^9 \cdots\cdots$$

$$\Gamma_2 = 6u^{10} + \frac{201}{2}u^{11} \cdots\cdots$$

$$\Gamma_3 = u^{12}$$

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$$+f^{12}$$

$$-\ 6f^{13} \qquad \text{(A·12)}$$

$$(1-M^2)\}\,].$$

$$-18u^{13}\cdots\cdots \qquad \text{(A·13)}$$

$$+130u^{14}+\cdots\cdots$$

$$-18u^{13}\cdots\cdots \qquad \text{(A·14)}$$

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Effects of Virtual Nucleon Pairs to the Electromagnetic Structure of the Nucleon

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Effects of virtual nucleon pairs to the electromagnetic structure of the nucleon are calculated covariantly in the charge independent pseudoscalar meson theory. A rigorous expression for the vertex operator is derived in § 2, which is strongly reminiscent of the old-fashioned first- and second-order perturbation theory. Our method of approximation is to replace the complete set of eigenfunctions of the total Hamiltonian by some suitable functions, which are chosen in this paper so as to take into account the main contribution from the virtual nucleon pairs. These effects are proved, regardless of the magnitude of the coupling constant, to reveal itself only through the renormalization factors, which affect the relative magnitudes of the isotopic scalar and vector parts of the nucleon-current contributions and the meson-current contributions. Numerical results are almost the same with those obtained by cutting off the momentum integration at the nucleon mass in the second order perturbation calculation both for the ratio of the anomalous magnetic moments of the proton and the neutron and the mean-square radii of the charge distributions.

§ 1. Introduction

Recently, high energy electron-nucleon scattering experiments have raised some important problems concerning the structure of the nucleon[1]. The fact that the mean-square radius of the charge distribution of the proton is rather large while that of the neutron is almost zero, which seems to have been established experimentally[2], indicate decisively the inadequacy of the static pion field theory even as the first approximation, since it gives equal and opposite radii of charge distribution for proton and neutron respectively.

There are also several other low energy phenomena, which have not yet been given satisfactory explanations and are beyond the scope of the static pion field theory; i. e., anomalous magnetic moment of the nucleon, S-wave pion-nucleon scattering and so on.*

In this paper we investigate the effects of virtual nucleon pairs on the electro-

[1] After the completion of this work, several works on the dispersion theoretic approach to the S-wave pion-nucleon scattering have been published. See, G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev. **106** (1957) 1337, and K. Tanaka, Phys. Rev. **108** (1957) 1629.

magnetic structure of the nucleon. We shall not take into consideration the contributions of the strange particles in this paper; since it seems desirable to establish how to treat the effects of virtual nucleon pairs for the evaluation of the contribution of the strange particles also. So we investigate, on the basis of the relativistic pseudoscalar meson theory, the effects of virtual nucleon pairs to the anomalous magnetic moments and the mean-square radii of the charge and magnetic moment distributions of the nucleons. Although there is a possibility that some of the above mentioned phenomena might indicate the limit of the applicability of the present local field theory, it will be necessary, even if it were the case, to deduce from the present field theory some critical results rigorous enough to establish the inconsistency between the present quantum field theory on one hand and the experimental results on the other.

We shall derive in § 2 an expression for the electromagnetic vertex operator of the nucleon, which is correct up to the first order in the electronic charge but is otherwise rigorous, and is not specialized to the kind of the virtual field and the type of interactions. The equation may also be written in a form strongly reminiscent of the old-fashioned first and second order perturbation theory.

In § 3 our method of approximation will be presented, which consists in substituting the complete set of eigenstates of total Hamiltonian by a suitable approximate one.

In § 4 approximate calculations are performed, in § 5 the renormalization constants are evaluated and in § 6 numerical results are given for the anomalous magnetic moments and the mean-square radii of charge and magnetic moment distribution of the nucleon.

In § 7 our method of approximation and results are critically discussed.

§ 2. General expression for the vertex operator

We will start from the following equation, which may be taken as a defining equation for the vertex operator, $\Gamma_\nu(x'y';\xi')$:

$$\begin{aligned}&-\langle T\psi(x)\bar{\psi}(y)A_\mu(\xi)\rangle_0\\&=ie\mathcal{G}_{\mu\nu}(\xi\xi')G(xx')\Gamma_\nu(x'y';\xi')G(y'y)\,,\end{aligned}\tag{2·1}$$

where

$$\begin{aligned}G(xx')&=i\langle T\psi(x)\bar{\psi}(x')\rangle_0\,,\\\mathcal{G}_{\mu\nu}(\xi\xi')&=i\langle TA_\mu(\xi)A_\nu(\xi')\rangle_0\,,\end{aligned}$$

ψ and A_μ are Heisenberg operators for the nucleon and the photon fields respectively, and $\langle T\cdots\rangle_0$ means the true vacuum expectation value of Wick's T-product[3]. Summations or integrations should always be taken on the repeated suffices or variables.

Although all the formulae in this section hold both for renormalized and unrenormalized quantities, we shall use the renormalized quantities throughout this paper.

Operating on the left side of eq. (2·1) $\mathcal{D}_x=\gamma\partial_x+M$ and $\Box_\xi^2$ to the left and $\mathcal{D}_y=\gamma\partial_y+M$ to the right, we obtain

$$
\begin{aligned}
&-\mathcal{D}_x \Box_\xi^2 \langle T\psi(x)\bar{\psi}(y) A_\mu(\xi)\rangle_0 \mathcal{D}_y \\
&=\langle T\eta(x)\bar{\eta}(y) j_\mu(\xi)\rangle_0 - ie Z_2 \tau_p \gamma_\mu \delta(x-\xi)\delta(\xi-y) \\
&+ie\delta(x-\xi)\tau_p\gamma_\mu\{G(xy')\sum{}^*(y'y)-G(xy)\delta M\} \\
&+ie\delta(\xi-y)\{\sum{}^*(xx')G(x'y)-\delta M G(xy)\}\tau_p\gamma_\mu ,
\end{aligned}
\tag{2·2}
$$

where

$$
\begin{aligned}
\eta(x) &= \mathcal{D}_x \psi(x) , \\
\bar{\eta}(y) &= \bar{\psi}(y)\mathcal{D}_y , \\
j_\mu(\xi) &= -\Box_\xi^2 A_\mu(\xi)
\end{aligned}
\tag{2·3}
$$

and use is made of relations

$$
\begin{aligned}
&\delta(\xi_0-y_0)[j_\mu(\xi)\bar{\psi}, (y)\gamma_4] = ie\delta^4(\xi-y)\bar{\psi}(\xi)\tau_p\gamma_\mu , \\
&\delta(x_0-\xi_0)[\gamma_4\psi(x), j_\mu(\xi)] = ie\delta^4(x-\xi)\tau_p\gamma_\mu\psi(\xi) .
\end{aligned}
$$

and

$$
-igZ_1Z_2^{-1}\gamma_5\tau_i\langle T\phi_i(x)\psi(x)\bar{\psi}(y)\rangle_0 = i\sum{}^*(xx')G(x'y) .
$$

Multiplying the free nucleon wave functions $\bar{u}(x)$ and $u(y)$, which satisfy

$$
\bar{u}(x)\mathcal{D}_x = 0 \quad \text{and} \quad \mathcal{D}_y u(y) = 0
$$

from the left and right of eq. (2·2) respectively, and using

$$
\begin{aligned}
&\{G(xy')\sum{}^*(y'y)-G(xy)\delta M\}u(y) \\
&\quad = G(xy')Z_2[\delta(y'-y)\mathcal{D}_y]u(y) = Z_2\delta(x-y)u(y)
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{u}(x)\{\sum{}^*(xx')G(x'x)-\delta M G(xy)\} \\
&\quad = \bar{u}Z_2[\mathcal{D}_x\delta(x-x')]G(x'y) = Z_2\bar{u}(x)\delta(x-y),
\end{aligned}
$$

one obtains

$$
\begin{aligned}
&-\bar{u}(x)\mathcal{D}_x\Box_\xi^2\langle T\psi(x)\bar{\psi}(y)A_\mu(\xi)\rangle_0\mathcal{D}_y u(y) \\
&\quad -ieZ_2\bar{u}(x)\tau_p\gamma_\mu u(x)\delta(x-\xi)\delta(\xi-y) \\
&\quad +\bar{u}(x)\langle T\eta(x)\bar{\eta}(y)j_\mu(\xi)\rangle_0 u(y).
\end{aligned}
\tag{2·4}
$$

We porform the same operation as above to the right side of eq. (2·1) to obtain

$$
ie\bar{u}(x)\Gamma'_\mu(xy;\xi)u(y)
$$

up to the first order in e. Then we obtain

$$
\begin{aligned}
&ie\bar{u}(x)\Gamma'_\mu(xy;\xi)u(y) \\
&\quad = ieZ_2\bar{u}(x)\tau_p\gamma_\mu u(y)\delta(x-\xi)\delta(\xi-y)
\end{aligned}
$$

$$+\bar{u}(x)\langle T\eta(x)\bar{\eta}(y)j_\mu(\xi)\rangle_0 u(y). \tag{2·5}$$

This is our fundamental equation. One may also derive this equation starting from the S-matrix in the interaction representation for the scattering of a nucleon by an external electromagnetic field, as was done by Low[4] for the case of boson-fermion scattering.

One may also derive the following equation from eq. (2·5), which may be of some use. Choosing plane wave states with momenta p' and p as the wave functions $\bar{u}(x)$ and $u(y)$ respectively and expressing the space-time dependence of Heisenberg operators, $O_H(x)=\exp(-iPx)O_H(0)\exp(iPx)$, where P_μ is the total four momentum operator of the meson-nucleon system, explicitly, we integrate over the variables x, y and ξ after multiplying $\exp(ik\xi)$ to the both sides of eq. (2·4), to obtain

$$\begin{aligned}
&ie\bar{u}(p')\Gamma_\mu(p',p)u(p)-ie Z_2\bar{u}(p')\tau_p\gamma_\mu u(p)\\
&=+\sum_{\substack{n\\ p_n=p'}}{\sum_{\substack{m\\ p_m=p}}}{}'\frac{(\Psi_0,V_{p'}^\dagger\Psi_n)(\Psi_n,j_\mu\Psi_m)(\Psi_m,V_p\Psi_0)}{(E_n-p_0'-i\epsilon)(E_m-p_0-i\epsilon)}\\
&\quad-\sum_{\substack{n\\ p_n=-p}}{\sum_{\substack{m\\ p_m=-p'}}}{}'\frac{(\Psi_0,V_p\Psi_n)(\Psi_n,j_\mu\Psi_m)(\Psi_m,V_{p'}^\dagger\Psi_0)}{(E_n+p_0-i\epsilon)(E_m+p_0'-i\epsilon)}\\
&\quad+\sum_{\substack{n\\ p_n=p'}}\sum_{\substack{m\\ p_m=k}}\frac{(\Psi_0,V_{p'}^\dagger\Psi_n)(\Psi_n,V_p\Psi_m)(\Psi_m,j_\mu\Psi_0)}{(E_n-p_0'-i\epsilon)(E_m-k_0-i\epsilon)}\\
&\quad-\sum_{\substack{n\\ p_n=-p}}\sum_{\substack{m\\ p_m=k}}\frac{(\Psi_0,V_p\Psi_n)(\Psi_n,V_{p'}^\dagger\Psi_m)(\Psi_m,j_\mu\Psi_0)}{(E_n+p_0-i\epsilon)(E_m-k_0-i\epsilon)}\\
&\quad+\sum_{\substack{n\\ p_n=-k}}\sum_{\substack{m\\ p_m=p}}\frac{(\Psi_0,j_\mu\Psi_n)(\Psi_n,V_{p'}^\dagger\Psi_m)(\Psi_m,V_p\Psi_0)}{(E_n+k_0-i\epsilon)(E_m-p_0-i\epsilon)}\\
&\quad-\sum_{\substack{n\\ p_n=-k}}\sum_{\substack{m\\ p_m=-p'}}\frac{(\Psi_0,j_\mu\Psi_n)(\Psi_n,V_p\Psi_m)(\Psi_m,V_{p'}^\dagger\Psi_0)}{(E_n+k_0-i\epsilon)(E_m+p_0'-i\epsilon)},
\end{aligned}\tag{2·6}$$

where $k_\mu=p_\mu'-p_\mu$ and

$$V_{p'}^\dagger(x)=\bar{u}(p')\eta(x)=Z_2^{-1}[H'(x),a(p')],$$

$$V_p(y)=\bar{\eta}(y)u(p)=Z_2^{-1}[a^*(p),H'(y)],$$

$a(p')$ and $a^*(p)$ are the annihilation and the creation operators for a nucleon with momenta p' and p respectively, H' is the interaction Hamiltonian density, and Ψ_n's are the complete set of the eigenfunctions of the total Hamiltonian belonging to eigenvalue E_n. In eq. (2·6) j_μ, V_p and $V_{p'}^\dagger$ mean corresponding operators at the origin of the space-time respectively. Apparently eq. (2·6) has the form strongly reminiscent of the old-fashioned first- and second-order perturbation theory although there are following differences: (a) Every possible intermediate state Ψ_n which is allowed by the conservation laws will enter into the sums. (b) The matrix elements must be calculated between exact eigenstates

of the Hamiltonian rather than between first approximations to these eigenstates. One may also define a suitable analytic continuation to the complex energy plane of eq. (2·6) analogously to the case of static pion-nucleon seattering[7]. It is most convenient, for this purpose, to take a coordinate system where $k_i=0$, which is the center of mass system in the case of electron-nucleon scattering. We define $I^{\mu}_{\gamma'\gamma}(\boldsymbol{p}', \boldsymbol{p}; z)$ by the quantity obtained by replacing $V^{\dagger}_{\nu'}$, V_{ν} and $p_0=p_{0'}$ by $V_{\gamma'}$, V_{γ} and a complex variable z respectively, in the right side of eq. (2·6). Then, $I^{\mu}_{\gamma'\gamma}(\boldsymbol{p}', \boldsymbol{p}; z)$ has trivial q- and q'-dependences and satisfies the following relations:

$$I^{\mu}_{q'q}(\boldsymbol{p}', \boldsymbol{p}; z)=I^{\mu\dagger}_{qq'}(\boldsymbol{p}, \boldsymbol{p}'; z^*) \tag{2·7}$$

$$\lim_{z\to E_{p'}+i\varepsilon} I^{\mu}_{p'p}(\boldsymbol{p}', \boldsymbol{p}; z)$$

$$=ie\bar{u}(p')\Gamma_{\mu}(p', p)u(p)-ie\mathcal{Z}_2\bar{u}(p')\tau_p\gamma_{\mu}u(p). \tag{2·8}$$

One may use these relations to evaluate the electromagnetic properties of the nucleon approximately together with some approximations such as neglect of higher configurations in the summation over complete set in eq. (2·6). We do not, however, attempt such an approach in this paper.

§ 3. Approximation method

We calculate the effects of virtual nucleon pairs to the electromagnetic structure of the nucleon starting from eq. (2·5) or eq. (2·6). Our approximation is to replace in these equations the eigenfunctions of the total Hamiltonian, H, by some approximate functions, which should be chosen so as to take into account properly the main dynamical properties of the system.

Now we assume the following Lagrangian for the nucleon-pion-photon system:

$$\begin{aligned}\mathcal{L}=&-\tfrac{1}{4}\mathcal{Z}_2\{[\bar{\psi}, (\gamma(\partial-ie\tau_p A)+M)\psi]\\&-[(\gamma^T(\partial+ie\tau_p A)-M)\bar{\psi}, \psi]\}\\&-\tfrac{1}{2}\mathcal{Z}_3\{\{(\partial_\mu+ieA_\mu)\varphi^*, (\partial_\mu-ieA_\mu)\varphi\}\\&+\mu^2(\varphi^*\varphi+\varphi\varphi^*)+(\partial_\mu\phi_3)^2+\mu^2\phi_3^2\}\\&-\tfrac{1}{2}F_{\mu\nu}^2-\frac{i}{2}g\mathcal{Z}_1[\bar{\psi}, \gamma_5\tau_i\phi_i\psi]\\&+\tfrac{1}{2}\mathcal{Z}_2\delta M[\bar{\psi}, \psi]+\tfrac{1}{2}\mathcal{Z}_3\delta\mu^2\phi_i^2+\tfrac{1}{4}\mathcal{Z}_3^2\lambda\phi_i^2\phi_j^2,\end{aligned} \tag{3·1}$$

where $\varphi=(\phi_1-i\phi_2)/\sqrt{2}$ is the charged pion field operator, and the renormalization constant of the photon field ($\mathcal{Z}_3^{\text{photon}}$) is put equal to 1, since our considerations are restricted to the first order in e. Then, $j_\mu(\xi)$, $\gamma_i(x)$ and $\bar{\gamma}_i(y)$, defined by eqs. (2·3), are given by

$$j_\mu(\xi)=\frac{ie}{2}\mathcal{Z}_2[\bar{\psi}, \tau_p\gamma_\mu\psi]-\frac{e}{2}\mathcal{Z}_3\phi_i T_{ij}\partial_\mu\phi_j,$$

$$\eta(x) = -ig Z_1 Z_2^{-1} \gamma_5 \tau_i \phi_i \psi + \delta M \psi,$$
$$\bar{\eta}(y) = -ig Z_1 Z_2^{-1} \bar{\psi} \gamma_5 \tau_i \phi_i + \delta M \bar{\psi}, \tag{3·2}$$

where

$$T_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We will assume in this paper the following simple forms for the approximate state vectors as substitutes for the true state vectors;

$$\Psi_n \to \exp(-iS)\Phi_n, \tag{3·3}$$

$$S = \int \bar{\psi} \gamma_\lambda \gamma_5 \tau_i \phi_i \frac{w[\phi]}{\phi} \psi \, d\sigma_\lambda, \tag{3·4}$$

where $\phi = (\phi_i^2)^{1/2}$, Φ_n's are the eigenfunctions of the free Hamiltonian, H_0, with the same eigenvalues and boundary conditions as Ψ_n's, i. e.,

$$H\Psi_n = E_n \Psi_n,$$
$$H_0 \Phi_n = E_n \Phi_n, \tag{3·5}$$

(we assume that bound states do not exist) and unknown functional $w[\phi]$ should be determined so as to make $\exp(iS) H \exp(-iS)$ as diagonal as possible.

We will fix $w[\phi]$ by the requirement that the expectation values of the total Hamiltonian with respect to the approximate eigenstates, $\exp(-iS)\Phi_n$, should be minimum:

$$\delta(\exp(-iS)\Phi_n, H\exp(-iS)\Phi_n) = 0. \tag{3·6}$$

$w[\phi]$ is then given by

$$w[\phi(x)] = \tfrac{1}{2} \tan^{-1} \frac{g_0 \phi(x)}{M_0} \tag{3·7}$$

where

$$g_0 = g Z_1 Z_2^{-1}, \qquad M_0 = M - \delta M,$$

as was shown by Sawada[6]. When eq. (3·7) is substituted in eq. (3·4), $\exp(-iS)$ coincides exactly with the Foldy transformation function[7], although eq. (3·3) is, in our case, not a unitary transformation but an approximate representation of the "cloud" of the physical nucleon.

Although approximation (3·3) is indeed a very rough one as will be easily seen by eq. (3·8) below, it does include the main contribution of virtual nucleon pairs in the charge independent pseudoscalar meson theory. One may see this circumstance by observing that the matrix element for the creation or annihilation of a nucleon pair of

$$\exp(iS) H \exp(-iS) = H_0 + \frac{g Z_1 Z_2^{-1}}{2M} \int \bar{\psi} \gamma_5 \gamma_\mu \partial_\mu (\tau_i \phi_i) \psi \, dx$$

$$+\frac{g^2}{2M}\int\bar{\psi}\psi\phi_i^2 d\boldsymbol{x}+\frac{g^2}{(2M)^2}\int\bar{\psi}(\boldsymbol{\tau}\cdot[\boldsymbol{\phi}\times\boldsymbol{\pi}])\psi d\boldsymbol{x}+\cdots \tag{3·8}$$

is very small compared with that of

$$H=H_0+igZ_1Z_2^{-1}\int\bar{\psi}\gamma_5\tau_i\phi_i\psi d\boldsymbol{x}. \tag{3·9}$$

Although the second and the following terms in the right side of eq. (3·8) are not included in our approximation (3·3), the contributions to the vertex operator of these terms are taken into account at least up to the second order in g by the structure of eq. (2·5) or eq. (2·6). Thus we calculate the vertex operator using the approximation (3·3) in spite of the well-known insufficiency of the perturbation theoretic treatment of the above-mentioned terms, since our aim in this paper is to see the effects of virtual nucleon pairs. Moreover one may easily take into account the contributions from the *P*-wave pion cloud in the static approximation by replacing the static part of our results by that obtained using the Chew-Low type or the dispersion theoretic approach. It is to be noted that we may not say about the validity or the invalidity of the approximation (3·3) by inspecting eq. (3·8), since the degree of approximation is very much dependent on the structure of the equation where the approximation is being made.

Now let us make replacement (3·3) in the right side of eq. (2·6). One may rewrite them in a manifestly covariant form. The same result, however, can be far more easily derived starting from eq. (2·5) as follows.

We assume the existence of a unitary operator U, which connects the Ψ_n's and Φ_n's defined by eq. (3·5) in the following way

$$\Psi_n=U\Phi_n, \tag{3·10}$$

where bound states are assumed to be not existent. One may rewrite, then, the second term of the right side of eq. (2·5) as follows:

$$\begin{aligned}&\bar{u}(x)\,(\Psi_0,\ T\eta_j(x)\bar{\eta}_j(y)j_\mu(\hat{\xi})\Psi_0)u(y)\\&\quad=\bar{u}(x)\,(\Phi_0,\ T\eta_i(x)\eta_j(y)\boldsymbol{j}_\mu(\hat{\xi})\Phi_0)u(y)\ ,\end{aligned} \tag{3·11}$$

where

$$\eta_i(x)=U^{-1}\eta_i(x)U,\ \text{etc.}\,. \tag{3·12}$$

As will be seen by

$$\begin{aligned}\eta_i(x)&=U^{-1}\eta_i(x)U=U^{-1}e^{iHx_0}\eta_i(\boldsymbol{x},0)e^{-iHx_0}U\\&=e^{iHx_0}\boldsymbol{\eta}(\boldsymbol{x},0)e^{-iHx_0}=e^{iH_0x_0}\boldsymbol{\eta}(\boldsymbol{x},0)e^{-iH_0x_0},\end{aligned}$$

the time development of $\eta(x)$ is given by the free Hamiltonian. For actual calculation it is convenient to represent U by the operators at the origin of the time.

If one uses $\exp(-iS)$ in place of the above U, where S is given by eqs. (3·4) and (3·7), eq. (3·11) becomes the convariant expression of the corresponding terms of eq. (2·6) with the approximation (3·3), where

$$\eta(x) = -ig Z_1 Z_2^{-1} \gamma_5 \tau_i \phi_i(x) e^{-is(x)} \psi(x) + \delta M e^{-is(x)} \psi(x) ,$$

$$\bar{\eta}(y) = -ig Z_1 Z_2^{-1} \bar{\psi}(y) e^{-is(y)} \gamma_5 \tau_i \phi_i(y) + \delta M \bar{\psi}(y) e^{-is(y)} ,$$

$$j_\mu(\xi) = j_\mu^n(\xi) + j_\mu^m(\xi) ,$$

$$j_\mu^n(\xi) = ie Z_2 \bar{\psi}(\xi) \gamma_\mu \left\{ \tau_p - \left(\tau_3 - \phi_3 \frac{\tau_i \phi_i}{\phi^2} \right) \sin^2 w \right.$$

$$\left. - i\gamma_5 \frac{\tau_3 \tau_i \phi_i - \phi_3}{2\phi} \sin 2w \right\} \psi(\xi) ,$$

$$j_\mu^m(\xi) = -\frac{e}{2} Z_3 \phi_i T_{ij} \partial_\mu \phi_j , \tag{3·13}$$

and

$$s(x) = \gamma_5 \tau_i \phi_i w[\phi(x)]/\phi .$$

We have dropped the terms proportional to the normal of the spacelike surface since it does not contribute to the S-matrix.

We have obtained, after all, the following expression which we will evaluate in the next section :

$$ie\bar{u}(x) \Gamma_\mu(x\gamma\,;\xi) u(y)$$

$$= ie Z_2 \bar{u}(x) \tau_p \gamma_\mu u(y) \delta(x-\xi) \delta(\xi-y)$$

$$+ \bar{u}(x) (\Phi_0, T\eta(x)\bar{\eta}(y) j_\mu(\xi) \Phi_0) u(y), \tag{3·14}$$

where $\eta(x)$, $\bar{\eta}(y)$ and $j_\mu(\xi)$ are given by Eqs. (3·13).

§ 4. Calculation of (3·14)

The second term of the right side of eq. (3·14) may be evaluated straightforwardly using the prescription given by Kinoshita and Nambu and by Anderson[8], since the time-dependences of $\eta(x)$, $\bar{\eta}(y)$ and $j_\mu(\xi)$ are given by the free Hamiltonian.

For the meson current contribution the result is given by

$$(\Phi_0, T\eta(x)\bar{\eta}(y) j_\mu^m(\xi) \Phi_0)$$

$$= -\frac{ie}{8} Z_3 \tau_3 \gamma_5 S_F(x-y) \gamma_5 \Delta_F(x-\xi) \overleftrightarrow{\partial}_{\xi\mu} \Delta_F(\xi-y) \cdot I^m(x-y) , \tag{4·1}$$

where

$$I^m(x-y) = \int du\, dv\, a(u) a(v) \exp\left[-\tfrac{1}{4}(u^2+v^2) + 2(u\cdot v) \frac{\Delta_F(x-y)}{\Delta_F(0)} \right]$$

$$\times \left[1 - \tfrac{1}{6}(u^2+v^2) - \tfrac{1}{3}(u\cdot v) \frac{\Delta_F(x-y)}{\Delta_F(0)} + \frac{1}{24} [u\times v]^2 \left\{ 1 - \frac{\ddot{\Delta}_F(x-y)}{\ddot{\Delta}_F(0)} \right\} \right] \tag{4·2}$$

and

$$a(\boldsymbol{u}) = (2\pi)^{-3}\int d\boldsymbol{\lambda}\, A[\boldsymbol{\lambda}] e^{i\boldsymbol{u}\boldsymbol{\lambda}},$$

$$A[\boldsymbol{\lambda}] = g Z_1 Z_2^{-1} \cos w[\sqrt{\Delta_F(0)}\,\boldsymbol{\lambda}] + \delta M \phi^{-1} \sin w[\sqrt{\Delta_F(0)}\,\boldsymbol{\lambda}]$$

and bold-face letters $\boldsymbol{\lambda}$ and $\boldsymbol{u}$ represent vectors in the isospin space.

The nucleon current contributions are given by

$$\begin{aligned}
&(\Phi_0, T\psi(x)\bar{\psi}(y)\boldsymbol{j}_\mu^n(\xi)\Phi_0)\\
&= \frac{ie}{8} Z_2[\gamma_5 S_F(x-\xi)\gamma_\mu S_F(\xi-y)\gamma_5\{I_1^n + \tau_3 I_2^n\}\\
&+ S_F(x-\xi)\gamma_\mu S_F(\xi-y)\{I_2^n + \tau_3 I_1^n\}\\
&+ \tau_3\gamma_5 S_F(x-\xi)\gamma_\mu\gamma_5 S_F(\xi-y) I_3^n\\
&+ \tau_3 S_F(x-\xi)\gamma_\mu\gamma_5 S_F(\xi-y)\gamma_5 I_0^n],
\end{aligned} \tag{4·3}$$

where I_1^n, $\cdots$, I_0^n are very complicated functions of $\Delta_F^{-1}(0)\Delta_F(x-\xi)$, $\Delta_F^{-1}(0)\Delta_F(\xi-y)$ and $\Delta_F^{-1}(0)\Delta_F(x-y)$, the detailed forms of which are given in Appendix A.

As is shown in detail in Appendix B, only the zeroth order terms with respect to $\Delta_F^{-1}(0)\Delta_F(x-\xi)$, $\Delta_F^{-1}(0)\Delta_F(\xi-y)$ and $\Delta_F^{-1}(0)\Delta_F(x-y)$ contribute to the observable quantities. As a result of this fact, it turns out that the principal effect of virtual nucleon pairs to the electromagnetic structure of the nucleon in the charge independent *ps-ps* theory reveals itself as the change of the normalization caused by the virtual dissociation. This seems to be a characteristic property of the *ps-ps* theory, owing to which evaluation of I^m, I_1^n, $\cdots$, I_0^n becomes extremely easier.

At a first glance one might think that $\Delta_F^{-1}(0)\Delta_F(x-y)$ in the exponential factor in the integrand of $I^m(x-y)$ corresponds to the propagation of infinite number of pions from y to x, but this is not the case, as will be discussed in Appendix C.

Observing that $w[\phi]$ is contained in I^m and I^n only through the forms,

$$\cos^2 w[\lambda] = \{1 + (1 + g_0^2\lambda^2 M_0^{-2})^{-1/2}\}/2,$$

and

$$\sin^2 w[\lambda] = \{1 - (1 + g_0^2\lambda^2 M_0^{-2})^{-1/2}\}/2,$$

we assume that $g_0^2\lambda^2 M_0^{-2}$ may be regarded to be very large compared with 1. This approximation seems to be rather good by the following reasons. Observing that

$$\begin{aligned}
&-\tfrac{1}{2}\delta_{ij}\Delta_F'(x-y) \equiv (\Psi_0, T\phi_i(x)\phi_j(y)\Psi_0)\\
&\to (\Phi_0, e^{iS}T\phi_i(x)\phi_j(y)e^{-iS}\Phi_0)\\
&= (\Phi_0, T\phi_i(x)\phi_j(y)\Phi_0 = \tfrac{1}{2}\delta_{ij}\Delta_F(x-y),
\end{aligned}$$

one may infer $Z_3 = 1$ and $g_0 = g Z_1 Z_2^{-1}$ is exactly the bare coupling constant in our approximation, so g_0^2 would be greater than g^2. Further the bare mass M_0 would be smaller than the observed mass because of the contribution from the vacuum polarization. Accordingly we may assume

$$g_0^2 \lambda^2 M_0^{-2} \gg 1, \tag{4·4}$$

since the left side would be larger than one for most important values of λ even if we replace g_0^2 by $g^2 \approx 4\pi \times 15$ and M_0^{-2} by M^{-2}.

One may expect, then, that the correct results would be between the results obtained below and the ordinary second order perturbation results, since, as will be seen by Fig. 1,

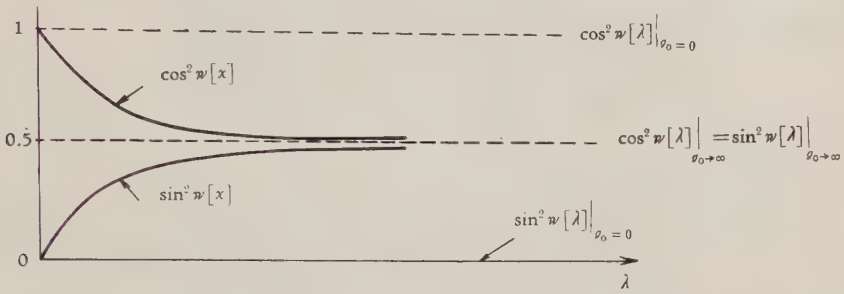

Fig. 1.

both $\cos^2 w[\lambda]$ and $\sin^2 w[\lambda]$ are monotonic functions of $g_0^2 \lambda^2 M_0^{-2}$ and their upper and lower bounds are given by their values obtained letting $g_0 \to 0$ and $g_0^2 \lambda^2 M_0^{-2} \to \infty$ respectively.

Adopting the above approximation (4·3), we may obtain I^m and I^n numerically using the values of the renormalization constants obtained in the next section.

§ 5. Renormalization

It is necessary to evaluate the renormalization constants to obtain the numerical results. This is performed in the following way.

If one has evaluated $S'(x-y) = i\langle \{\psi(x), \bar{\psi}(y)\} \rangle_0$, Z_2 and δM may be given by

$$Z_2^{-1} = -\frac{i}{4} \int \mathrm{Tr}[\gamma_4 S'(x)]_{x_0=0} d\boldsymbol{x}$$

and

$$\delta M = M - \frac{Z_2}{4} \int \mathrm{Tr}\left[\frac{\partial}{\partial x_0} S'(x)\right]_{x_0=0} d\boldsymbol{x}$$

respectively. These relations are easily derived if one expresses $S'(x)$ by the spectral functions[9]:

$$S'(x) = \int_0^\infty \{S(x\,;m)\rho_1(m^2) + \Delta(x\,;m)\rho_2(m^2)\}\, dm^2,$$

and observes that

$$Z_2^{-1} = \int_0^\infty \rho_1(m^2)\, dm^2$$

and

$$\delta M = Z_2 \int_0^\infty [(M-m)\rho_1 + \rho_2] dm^2.$$

It is, however, more appropriate for the contents of our approximation to evaluate S'_F than to evaluate S' for the determination of Z_2. In our approximation S'_F is given by

$$\begin{aligned}
-\tfrac{1}{2} S'_F(x-y) &= (\Psi_0, T\psi(x)\bar{\psi}(y)\Psi_0) \to (\Phi_0, T\psi(x)\bar{\psi}(y)\Phi_0) \\
&= -\tfrac{1}{2}\int du\, dv \exp\left[-\tfrac{1}{4}\left\{u^2+v^2+2(u\cdot v)\frac{\Delta_F(x-y)}{\Delta_F(0)}\right\}\right] \\
&\times\Big[S_F(x-y)b(u)b(v) \\
&-\tfrac{1}{2}\gamma_5 S_F(x-y)\gamma_5 c(u)c(v)\Big\{-\tfrac{1}{2}(u\cdot v)\Big(1-\frac{\Delta'_F(x-y)}{\Delta'_F(0)}\Big) \\
&+(3-\tfrac{1}{2}(u^2+v^2))\frac{\Delta_F(x-y)}{\Delta_F(0)}\Big\}\Big],
\end{aligned} \tag{5·1}$$

where

$$b(u) = (2\pi)^{-3}\int d\lambda\, e^{iu\lambda}\cos w[\sqrt{\Delta'_F(0)}\,\lambda],$$

and

$$c(u) = (2\pi)^{-3}\int d\lambda\, e^{iu\lambda}\frac{\sin w[\sqrt{\Delta'_F(0)}\,\lambda]}{\sqrt{\Delta'_F(0)}\,\lambda}.$$

As was mentioned before, only the terms of zeroth order in $\Delta'_F(0)\Delta_F(x-y)$ give non-zero contributions, so the second term in the bracket of eq. (5·1) disappears and eq. (5·1) becomes

$$S'_F(x-y) = Z_2 S_F(x-y), \tag{5·2}$$

where Z_2 is given by

$$Z_2 = \left[\int du\, \exp(-u^2/4)b(u)\right]^2. \tag{5·3}$$

This fact shows that in the pseudoscalar meson theory the main effect of the virtual nucleon pairs to the nucleon propagator is merely to alter the normalization. If, further, the approximation (4·4) is used, (5·3) gives

$$Z_2 = 1/2. \tag{5·4}$$

Z_1 may be obtained by evaluating $(\Psi_{p'}, (\Box_x^2 - \mu^2)\phi_i(x)\Psi_p)$, which is essentially the S-matrix element for the scattering of a nucleon with initial and final momenta p and p' respectively by an external meson field. Putting

$$O_i(\tilde{\xi}) \equiv (\Box^2-\mu^2)\phi_i(\tilde{\xi}) = \tfrac{1}{2} g Z_1 Z_3^{-1}[\bar{\psi}(\tilde{\xi}), \gamma_5 \tau_i \psi(\tilde{\xi})] - \delta\mu^2 \phi_i(\tilde{\xi}) + Z_3 \lambda \phi_i(\tilde{\xi}) \phi_j^2(\tilde{\xi}),$$

we obtain

$$\begin{aligned}(\Psi_{p'}, O_i(0)\Psi_p) &\to (\Phi_{p'}, e^{iS} O_i(0) e^{-iS} \Phi_p) \\ &= ig Z_1 Z_3^{-1} (\Phi_{p'}, \bar{\psi} e^{iS} \gamma_5 \tau_i e^{-iS} \psi \Phi_p) \\ &= ig Z_1 Z_3^{-1} (\bar{u}(p')\gamma_5 \tau_i u(p)) \{1 - \tfrac{2}{3}\langle \sin^2 w \rangle_0\}, \end{aligned} \tag{5·5}$$

where $u(p)$ is the usual Dirac spinor wave function. Using

$$\begin{aligned}(\Psi_{p'}, O_i(0)\Psi_p) &= (\Psi_{p'}, (\Box^2-\mu^2)\phi_i(\tilde{\xi})\Psi_p)|_{\xi=0} \\ &= -[(p_\lambda'-p_\lambda)^2+\mu^2] \Delta_F'(p'-p) ig \Gamma_5'(p', p) \tau_i Z_3^{1/2}, \end{aligned} \tag{5·6}$$

and remembering that in our approximation

$$-[(p_\lambda'-p_\lambda)^2+\mu^2]\Delta_F'(p'-p) = 1 \quad \text{and} \quad Z_3 = 1, \tag{5·7}$$

(5·5) should be equal to $ig(\bar{u}(p')\gamma_5\tau_i u(p))$ when p' approaches to p. Thus Z_1 is given, in our approximation, by

$$\begin{aligned} Z_1^{-1} &= 1 - \tfrac{2}{3}\langle \sin^2 w \rangle_0 \\ &= 1 - \tfrac{2}{3}(2\pi)^{1/2} \int_0^\infty \left\{1 - \left(1 + \frac{g_0^2 (\Delta^{(1)}(0))^2}{4M_0^2} y^2\right)\right\}^{-1/2} e^{-y^2/2} y^2 dy \\ &\fallingdotseq \tfrac{2}{3}. \end{aligned} \tag{5·8}$$

Approximation (4·4) is made in the last step of eq. (5·8).

δM is obtained by calculating the S matrix element for a nucleon propagating freely or by calculating directly the expectation value of the total Hamiltonian with respect to the physical one nucleon state in our approximation.

Both methods yield the same result

$$\begin{aligned}\delta M &= M_0 \left\{ \left\langle \left(1 + \frac{g_0^2 \phi^2}{M_0^2}\right)^{1/2} \right\rangle_0 - 1 \right\} \\ &\fallingdotseq 2 (g_0^2 \Delta_F(0)/2)^{1/2}, \end{aligned} \tag{5·9}$$

where the approximation (4·4) is used in the last step.

§ 6. Numerical results

Now we may evaluate I^m and I^n numerically, using the values of the renormalization constants obtained in the preceding section and making the approximation (4·4), to obtain

$$\begin{aligned} I^m &= g'^2, \\ I_1^n &= -\tfrac{2}{3} g'^2 \Delta_F(x-y), \\ I_2^n &= \tfrac{1}{6} g'^2 \Delta_F(x-y), \end{aligned}$$

$$I_3^n = I_4^n = I_5^n = I_6^n = 0 \,, \tag{6·1}$$

where

$$g'^2 = \tfrac{1}{2} Z_1^2 Z_2^{-2} \left(1 + \frac{8}{3\pi}\right)^2 g^2 = \tfrac{1}{2}\left(3 + \frac{8}{\pi}\right)^2 g^2 \,*. \tag{6·2}$$

All the divergences appearing in I^m and I^n are cancelled out by δM, and the vanishing of $I_3^n, \cdots, I_6^n$ is also the result of cancelling of δM-term with the other terms.

Vertex operator, eq. (3·14), may be written in the momentum space in the following form :

$$\Gamma_\mu(p', p) = Z_2' \tau_p \gamma_\mu + \gamma_\mu H_1(k^2) + \frac{1}{2M} \sigma_{\mu\nu} k_\nu H_2(k^2) \,, \tag{6·3}$$

where $k_\mu = p_\mu' - p_\mu$ and it is assumed that $\Gamma_\mu(p', p)$ is operating on $\bar{u}(p')$ and $u(p)$ on both sides. We have written Z_2 in the first term of the right side of eq. (6·3) as Z'_2, which should cancel the over-all divergence, to distinguish from Z_2 discussed in the preceding section, which should cancel the internal divergence and should be put equal to $\frac{1}{2}$ in our approximation.

Inserting eq. (6·1) into eqs. (4·1) and (4·3), $H_1(k^2)$ and $H_2(k^2)$ are given by

$$H_{1,2}(k^2) = Z_2 f_{1,2}^{(ns)}(k^2) + \tau_3 \frac{Z_2}{3} f_{1,2}^{(nv)}(k^2) + \tau_3 Z_3 f_{1,2}^{(m)}(k^2) \tag{6·4}$$

where $f_{1,2}^{(\ )}(k^2)$ are the ordinary second order perturbation results with the coupling constant g', and superscripts (ns), (nv) and (m) mean the isotopic scalar part of the nucleon current contribution, isotopic vector part of that and the meson current contribution, respectively. It is seen from eq. (6·4) that the effects of virtual nucleon pairs reduce the isotopic scalar and vector parts of the nucleon current contribution to 1 2 and 1 6, respectively, relative to the meson current contribution compared with the second order perturbation results, since $Z_2 = 1\ 2$ and $Z_3 = 1$ in eq. (6·4). Momentum dependent effects do not appear because of the vanishing of the terms including $J_\mu(0) J_\nu(x - \nu)$, etc, in (4·1) and (4·3). To determine Z_2', we divide $\Gamma_\mu(p', p)$ into the proton vertex, Γ_μ^p, and the neutron vertex, Γ_μ^n ;

$$\Gamma_\mu(p', p) = \tau_p \Gamma_\mu^p(p', p) + \tau_n \Gamma_\mu^n(p', p) \,,$$

where

$$\Gamma_\mu^p(p', p) = Z_2' \gamma_\mu + \gamma_\mu H_1^p(k^2) + \frac{1}{2M} \sigma_{\mu\nu} k_\nu H_2^p(k^2),$$

$$H_{1,2}^p(k^2) = Z_2 f_{1,2}^{(ns)}(k^2) + Z_2 f_{1,2}^{(nv)}(k^2) + Z_3 f_{1,2}^{(m)}(k^2),$$

* Which of the two coupling constants, $g^2/4\pi$ and $g'^2/4\pi$, should be identified with the Yukawa constant, ~ 0.08, is not clear in our calculation. One may decide it, for instance, by investigating pion-nucleon scattering in a similar way. Numerical results given below for the ratios of the anomalous magnetic moments of the proton and the neutron and of the mean square radii of the charge distributions of them are, however, not dependent on the value of the coupling constant.

$$\Gamma_\mu^n(p', p) = \gamma_\mu H_1^n(k^2) + \frac{1}{2M}\sigma_{\mu\nu}k_\nu H_2^n(k^2),$$

$$H_{1,2}^n(k^2) = Z_2 f_{1,2}^{(ns)}(k^2) - Z_2 f_{1,2}^{(nv)}(k^2) - Z_3 f_{11,2}^{(m)}(k^2).$$

By the requirement $\Gamma_\mu^p(p, p) = \gamma_\mu$, Z_2' is determined to be

$$Z_2' = 1 - H_1^p(0). \tag{6·5}$$

Further to satisfy the requirement that the total neutron charge should be zero, i. e. $\Gamma_\mu^n(p, p) = 0$, we modify $\Gamma_\mu^n(p', p)$ to

$$\Gamma_\mu^n(p', p) = \gamma_\mu\{H_1^n(k^2) - H_1^n(0)\} + \frac{1}{2M}\sigma_{\mu\nu}k_\nu H_2^n(k^2).$$

We may then write the renormalized vertices in the following form:

$$\Gamma_\mu^{p,n}(p', p) = \gamma_\mu F_1^{p,n}(k^2) + \frac{\mu^{p,n}}{2M}\sigma_{\mu\nu}k_\nu F_2^{p,n}(k^2), \tag{6·6}$$

where the anomalous magnetic moments $\mu^{p,n}$ are given by

$$\mu^{p,n} = H_2^{p,n}(0), \tag{6·7}$$

the form factor of the vector interaction by

$$F_1^{p,n}(k^2) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + H_1^{p,n}(k^2) - H_1^{p,n}(0), \tag{6·8}$$

and that of the tensor interaction by

$$F_2^{p,n}(k^2) = H_2^{p,n}(k^2)/H_2^{p,n}(0). \tag{6·9}$$

The numerical value of the ratio of the isotopic scalar part to the vector part of the anomalous magnetic momont becomes

$$\left|\frac{\mu^p + \mu^n}{\mu^p - \mu^n}\right| = 0.49. \tag{6·10}$$

This result is very large compared with the experimental value, 0.03. However, we should compare our results rather with the second order perturbation results than with the experimental values, since we have essentially calculated the corrections due to the virtual nucleon pairs to the second order perturbation calculation, as will be easily seen by the fact that letting $S = 0$ in eq. (3·3) one obtains the second order perturbation results. The value given by (6·10) is better than the second order perturbation result, 0.77, and almost the same with the value, 0.51, obtained by cutting off the momentum of the virtual meson in the second order perturbation result at the nucleon rest mass.[10]

Mean-square radius of the "charge" distribution, $\langle r^2\rangle_1^{p,n}$, may be obtained by

$$F_1^{p,n}(k^2) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \tfrac{1}{6}k^2\langle r^2\rangle_1^{p,n} + \cdots, \tag{6·11}$$

the numerical values of which are

$$\langle r^2\rangle_1^{(p,n)}=\frac{1}{\pi}\frac{g'^2}{4\pi}\frac{1}{M^2}\times\begin{cases}1.37\cdots\text{proton},\\ -0.84\cdots\text{neutron}.\end{cases}\tag{6·12}$$

The ratio of the above value for the proton to that for the neutron is worse than the ratio of the second order perturbation result:

$$\langle r^2\rangle_1^{(2)p,n}=\frac{3}{\pi}\frac{g^2}{4\pi}\frac{1}{M^2}\times\begin{cases}0.487,\\ -0.128,\end{cases}\tag{6·13}$$

and is almost the same with the results obtained by cutting off the momentum integration in the second order perturbation calculation at the momentum slightly smaller than M.[11]

The mean-square radius of the "magnetic moment" distribution, $\langle r^2\rangle_2^{p,n}$, may also be obtained in a similar manner as

$$\langle r^2\rangle_2^{p,n}=\begin{cases}9.5\ \mathrm{M}^{-2},\\ 3.9\ \mathrm{M}^{-2},\end{cases}\tag{6·14}$$

or

$$\langle r^2\rangle_2^{p,n}=\left\{\begin{matrix}1.01\\ \\ 1.08\end{matrix}\right\}\times10^{-26}\ \mathrm{cm}^2,\tag{6·15}$$

where in eq. (6·14) we have used the results obtained in this paper for the anomalous magnetic moments and in eq. (6·15) the experimental magnetic moments are used.

§ 7. Conclusion and discussions

We have evaluated the effects of virtual nucleon pairs to the electromagnetic structure of the nucleon in the charge independent pseudoscalar meson theory, starting from general formulae (2·5) and (2·6) for the vertex operator. As a substitute for the complete set of eigenfunctions of the total Hamiltonian, Ψ_n, we have used $\exp(-iS)\Psi_n$ given by (3·3), which is considered to include the main contributions from the virtual nucleon pairs in the pseudoscalar meson theory. This is our essential approximation. Vacuum expectation values of the T-product have then been evaluated, and the rather complicated results obtained in eqs. (4·1) and (4·3) could be extremely simplified by the fact that only the contributions from the zeroth order terms with respect to $J_\mu^*(0)J_\nu(x-\nu)$, etc., do not vanish. This fact seems to be due to the special character of the pseudoscalar meson theory. Therefore the effects of the virtual nucleon pairs to the vertex operators reveal itself essentially through the normalization factors which affect the relative ratios between the isotopic scalar and vector parts of the nucleon current and the meson current contributions, respectively.

In deriving the numerical results we have further used the approximation (4·4), which seems to be rather good as was discussed under eq. (4·4).

Final results are given by multiplying, by various constant factors, the isotopic scalar

and vector parts of the nucleon current contributions and the meson current contributions calculated by the second order perturbation, respectively. These circumstances will be similar in cases of other processes such as meson-meson scattering, γ-decay of the neutral pion, etc.

Numerical results are almost the same as that obtained by cutting off the momentum of the virtual meson in the second order perturbation calculation at the nucleon rest mass both for the anomalous magnetic moments and the mean-square radii of the charge and moment distributions.

Merits of our method would be that the calculation is Lorentz-covariant throughout and that the approximation is made only once in eq. (3·3) except in eq. (4·4) which may be justified, so that one may rather easily examine the validity or the insufficiency of our approximation.

Our approximate state vectors, (3·3), are of course very crude, and the contributions to the vertex operator from the P-wave pions have been taken into account only perturbation theoretically. Although one may easily take into account all the effects of the P-wave pions in the static approximation by replacing the static part of our calculation by that obtained using either Chew-Low approach or the dispersion theoretic one, it would be necessary for the inclusion of the correlation between the P-wave and the pair effects, which might be very important, to improve the approximation (3·3) so as to take the P-wave cloud into consideration dynamically. Such a problem has not been treated in the present paper.

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Appendix A Explicit forms of $I_1^n, \cdots, I_{6n}$ in eq. (4·3).

Functions $I_1^n, \cdots, I_6^n$ in Eq. (4·3) are given by the following equations:

$$I_i^n = \int d\boldsymbol{u}\, d\boldsymbol{v}\, d\boldsymbol{w} \exp\Big[-\tfrac{1}{4}(\boldsymbol{u}^2+\boldsymbol{v}^2+\boldsymbol{w}^2) - \tfrac{1}{2}\boldsymbol{u}\boldsymbol{v}\,\frac{\Delta_F(x-y)}{\Delta_F(0)} - \tfrac{1}{2}\boldsymbol{v}\boldsymbol{w}\,\frac{\Delta_F(y-\hat{\xi})}{\Delta_F(0)} - \tfrac{1}{2}\boldsymbol{w}\boldsymbol{u}\,\frac{\Delta_F(\hat{\xi}-x)}{\Delta_F(0)} \Big] \cdot J_i^n ,$$

where

$$\begin{aligned}
J_1^n &= -\tfrac{3}{2} A_2(\boldsymbol{u})\, A_2(\boldsymbol{v})\, \delta(\boldsymbol{w}) \{ \bar{\Delta}_F(x-y) \\
&\quad -\tfrac{1}{6}(\boldsymbol{u}^2+\boldsymbol{v}^2)\,\bar{\Delta}_F(x-y) - \tfrac{1}{6}\boldsymbol{u}\boldsymbol{v}(1+\bar{\Delta}_F^2(x-y)) \}, \\
J_2^n &= \tfrac{1}{2} A_2(\boldsymbol{u})\, A_2(\boldsymbol{v})\, \delta(\boldsymbol{w}) \{ \bar{\Delta}_F(x-y) - \tfrac{1}{6}(\boldsymbol{u}^2+\boldsymbol{v}^2)\,\bar{\Delta}_F(x-y) \\
&\quad -\tfrac{1}{6}\boldsymbol{u}\boldsymbol{v}(1+\bar{\Delta}_F^2(x-y)) \} \\
&\quad - A_2(\boldsymbol{u})\, A_2(\boldsymbol{v})\, B_1(\boldsymbol{w}) \{ \bar{\Delta}_F(x-y) - \tfrac{1}{6}(\boldsymbol{u}^2+\boldsymbol{v}^2)\,\bar{\Delta}_F(x-y) \\
&\quad -\tfrac{1}{6}\boldsymbol{w}^2\,\bar{\Delta}_F(x-\hat{\xi})\,\bar{\Delta}_F(\hat{\xi}-y) - \tfrac{1}{6}\boldsymbol{u}\boldsymbol{v}(1+\bar{\Delta}_F^2(x-y)) \\
&\quad -\tfrac{1}{6}\boldsymbol{v}\boldsymbol{w}(\bar{\Delta}_F(x-\hat{\xi}) + \bar{\Delta}_F(x-y)\,\bar{\Delta}_F(y-\hat{\xi}))
\end{aligned}$$

$$-\tfrac{1}{6}\boldsymbol{w}\boldsymbol{u}\,(\bar{J}_F(y-\xi)+\bar{J}_F(x-\xi)\,\bar{J}_F(x-y))\}$$

$$-\tfrac{1}{2}A_2(\boldsymbol{u})\,A_2(\boldsymbol{v})\,B_3(\boldsymbol{w})\,\{\bar{J}_F(x-\xi)\,\bar{J}_F(\xi-y)\Big(6-(\boldsymbol{u}^2+\boldsymbol{v}^2+2\boldsymbol{w}^2)$$

$$-\tfrac{5}{3}\boldsymbol{u}\boldsymbol{v}\,\bar{J}_F(x-y)-2\,\boldsymbol{v}\boldsymbol{w}\,\bar{J}_F(\xi-y)-2\,\boldsymbol{w}\boldsymbol{u}\,\bar{J}_F(x-\xi)+\frac{\boldsymbol{w}^2}{6}\Big)$$

$$-\bar{J}_F(x-y)\,(1+\tfrac{5}{6}\boldsymbol{u}^2\bar{J}_F^2(x-\xi)+\tfrac{5}{6}\boldsymbol{v}^2\bar{J}_F^2(y-\xi)$$

$$-\tfrac{1}{6}(\boldsymbol{u}^2+\boldsymbol{v}^2+\boldsymbol{w}^2))-\boldsymbol{u}\boldsymbol{v}\,(\tfrac{5}{6}\bar{J}_F^2(x-\xi)+\bar{J}_F^2(y-\xi)-\tfrac{1}{6})$$

$$-\tfrac{1}{6}\boldsymbol{v}\boldsymbol{w}\,(5\,\bar{J}_F(x-\xi)+3\,\bar{J}_F(x-y)\,\bar{J}_F(y-\xi))-\tfrac{1}{6}\boldsymbol{w}\boldsymbol{u}\,(5\,\bar{J}_F(y-\xi)$$

$$+3\,\bar{J}_F(\xi-x)\,\bar{J}_F(x-y))$$

$$+\frac{1}{12}[2\,\{(\boldsymbol{u}^2+\boldsymbol{v}^2)\,\bar{J}_F(x-\xi)-\boldsymbol{v}^2\bar{J}_F(x-y)\,\bar{J}_F(y-\xi)$$

$$+\boldsymbol{u}\boldsymbol{v}\,(\bar{J}_F(y-\xi)+\bar{J}_F(x-\xi)\,\bar{J}_F(x-y))+\boldsymbol{v}\boldsymbol{w}\,(\bar{J}_F(x-y)$$

$$+\bar{J}_F(x-\xi)\,\bar{J}_F(\xi-y))+\boldsymbol{w}\boldsymbol{u}\,(1+\bar{J}_F^2(x-\xi))\}$$

$$\times\{\boldsymbol{u}^2\bar{J}_F(x-y)\,\bar{J}_F(x-\xi)+(\boldsymbol{v}^2+\boldsymbol{w}^2)\,\bar{J}_F(y-\xi)$$

$$+\boldsymbol{u}\boldsymbol{v}\,\bar{J}_F(x-\xi)+\bar{J}_F(x-y)\,\bar{J}_F(y-\xi))+\boldsymbol{v}\boldsymbol{w}\,(1+\bar{J}_F^2(y-\xi))$$

$$+\boldsymbol{w}\boldsymbol{u}\,(\bar{J}_F(x-y)+\bar{J}_F(y-\xi)\,\bar{J}_F(\xi-x))$$

$$-\{\boldsymbol{u}^2\bar{J}_F^2(x-\xi)+\boldsymbol{v}^2\bar{J}_F^2(y-\xi)+\boldsymbol{w}^2$$

$$+2\,\boldsymbol{u}\boldsymbol{v}\,\bar{J}_F(x-\xi)\,\bar{J}_F(\xi-y)+2\,\boldsymbol{v}\boldsymbol{w}\,\bar{J}_F(y-\xi)$$

$$+2\,\boldsymbol{w}\boldsymbol{u}\,\bar{J}_F(x-\xi)\}\times\{(\boldsymbol{u}^2+\boldsymbol{v}^2)\,\bar{J}_F(x-y)$$

$$+\boldsymbol{w}^2\bar{J}_F(x-\xi)\,\bar{J}_F(\xi-y)+\boldsymbol{u}\boldsymbol{v}\,(1+\bar{J}_F^2(x-y))$$

$$+\boldsymbol{v}\boldsymbol{w}\,(\bar{J}_F(x-\xi)+\bar{J}_F(x-y)\,\bar{J}_F(y-\xi))$$

$$+\boldsymbol{w}\boldsymbol{u}\,(\bar{J}_F(\xi-y)+\bar{J}_F(x-\xi)\,\bar{J}_F(x-y))\},$$

$$J_3^n=A_1(\boldsymbol{u})\,A_1(\boldsymbol{v})\,\delta(\boldsymbol{w}),$$

$$J_4^n=A_1(\boldsymbol{u})\,A_1(\boldsymbol{v})\,\{\delta(\boldsymbol{w})-2\,B_1(\boldsymbol{w})$$

$$+B_3(\boldsymbol{w})[1-\tfrac{1}{6}(\boldsymbol{u}^2\bar{J}_F^2(x-\xi)+\boldsymbol{v}^2\bar{J}_F^2(y-\xi)$$

$$+2\,\boldsymbol{u}\boldsymbol{v}\,\bar{J}_F(x-\xi)\,\bar{J}_F(\xi-y))]\},$$

$$J_5^n=2\,A_2(\boldsymbol{u})\,A_1(\boldsymbol{v})\,B_2(\boldsymbol{w})[\bar{J}_F(x-\xi)$$

$$-\tfrac{1}{6}(\boldsymbol{u}^2+\boldsymbol{w}^2)\,\bar{J}_F(x-\xi)-\tfrac{1}{6}\boldsymbol{v}^2\bar{J}_F(x-y)\,\bar{J}_F(\xi-y)$$

$$-\tfrac{1}{6}\boldsymbol{w}\boldsymbol{u}\,(1+\bar{J}_F^2(x-\xi))-\tfrac{1}{6}\boldsymbol{v}\boldsymbol{w}\,(\bar{J}_F(x-y)$$

$$+\bar{J}_F(x-\xi)\,\bar{J}_F(\xi-y))-\tfrac{1}{6}\boldsymbol{u}\boldsymbol{v}\,(\bar{J}_F(y-\xi)$$

$$+\bar{J}_F(y-x)\,\bar{J}_F(x-\xi))],$$

$$
\begin{aligned}
J_{6'}^{n} = & -2A_1(\boldsymbol{u})A_2(\boldsymbol{v})B_2(\boldsymbol{w})[\bar{\Delta}_F(\hat{\xi}-y) \\
& -\tfrac{1}{6}(\boldsymbol{v}^2+\boldsymbol{w}^2)\bar{\Delta}_F(\hat{\xi}-y) - \tfrac{1}{6}\boldsymbol{u}^2\bar{\Delta}_F(x-y)\bar{\Delta}_F(\hat{\xi}-x) \\
& -\tfrac{1}{6}\boldsymbol{v}\boldsymbol{w}(1+\bar{\Delta}_F^{2}(y-\hat{\xi})) - \tfrac{1}{6}\boldsymbol{w}\boldsymbol{u}(\bar{\Delta}_F(x-y) \\
& +\bar{\Delta}_F(y-\hat{\xi})\bar{\Delta}_F(\hat{\xi}-x)) - \tfrac{1}{6}\boldsymbol{u}\boldsymbol{v}(\bar{\Delta}_F(x-\hat{\xi}) \\
& +\bar{\Delta}_F(x-y)\bar{\Delta}_F(y-\hat{\xi}))],
\end{aligned}
$$

where $\bar{\Delta}_F(x-y) = \Delta_F^{-1}(0)\Delta_F(x-y)$ etc. and $A_i(\boldsymbol{u})$ and $B_i(\boldsymbol{u})$ are given by

$$
A_i[(\Delta_F(0))^{1/2}\phi(x)] \equiv (2\pi)^{-3}\int d\boldsymbol{u}\exp(-i\phi(x)\boldsymbol{u})A_i(\boldsymbol{u}),
$$

$$
A_1[\phi(x)] \equiv -gZ_1Z_2^{-1}\phi\sin w[\phi] + \delta M\cos w[\phi],
$$

$$
A_2[\phi(x)] \equiv gZ_1Z_2^{-1}\cos w[\phi] + \delta M\phi^{-1}\sin w[\phi],
$$

$$
B_1[\phi(x)] \equiv \sin^2 w[\phi],
$$

$$
B_2[\phi(x)] \equiv (2\phi)^{-1}\sin 2w[\phi],
$$

$$
B_3[\phi(x)] \equiv \phi^{-2}\sin^2 w[\phi].
$$

and $w[\phi]$ is given by Eq. (3·7).

Appendix B

It will be shown in this Appendix B that only the zeroth order terms with respect of $\Delta_F^{-1}(0)\Delta_F(x-y)$, $\Delta_F^{-1}(0)\Delta_F(x-\hat{\xi})$ and $\Delta_F^{-1}(0)\Delta_F(\hat{\xi}-y)$ in I^m, I_1^n, $\cdots$, I_6^n in Eqs. (4·2) and (4·3) give finite contributions to the observable quantities.

We consider the meson current contribution to the vertex operator, but the following considerations are also applicable to the nucleon current contribution.

Fourier transform of the meson current part of the vertex operator, $\Gamma_\mu'^{(m)}(p', p)$, is given by

$$
\begin{aligned}
& (2\pi)^{-4}\delta(p'-p-k)\Gamma_\mu'^{(m)}(p', p) \\
& \quad = \int dxdyd\hat{\xi}\, e^{-ip'x+ipy+ik\hat{\xi}}\Gamma_\mu'^{(m)(2)}(x, y; \hat{\xi})I^m(x-y), \qquad \text{(B·1)}
\end{aligned}
$$

where

$$
\begin{aligned}
& g^2Z_1^{2}Z_2^{-2}\Gamma_\mu'^{(m)(2)}(x, y; \hat{\xi}) \\
& \quad = -\frac{g^2}{8}Z_1^{2}Z_2^{-2}Z_3\tau_3\gamma_5 S_F(x-y)\gamma_5\Delta_F(x-\hat{\xi})\overleftrightarrow{\partial}_{\hat{\xi}\mu}\Delta_F(\hat{\xi}-y) \qquad \text{(B·2)}
\end{aligned}
$$

is the second order perturbation result and $I^m(x-y)$ is given by eq. (4·2).

Using an identity

$$I^m(x-y)=I^m(y-x)=\int_{-\infty}^{\infty}\frac{d^4a}{(2\pi)^4}\int d^4L e^{iL(a+x-y)}I^m(a)\,,$$

one obtains from eq. (B·1),

$$\Gamma_\mu^{(m)}(p',\ p)=\int_{-\infty}^{\infty}\frac{d^4a}{(2\pi)^4}\int d^4L I^m(a)e^{iLa}\Gamma_\mu^{(m)(2)}(p'-L,\ p-L)\,. \tag{B·3}$$

Expanding $I^m(a)$ to the powers of $\Delta_F^{-1}(0)\Delta_F(a)$,

$$I^m(a)=\sum_{i=0}^{\infty}I_i^m\left[\frac{\Delta_F(a)}{\Delta_F(0)}\right]^{2i} \tag{B·4}$$

(odd powers of $\Delta_F^{-1}(0)\Delta_F(a)$ vanish by the integration over u and $\mathbf{r}$ in eq. (4·2)), one may write eq. (B·3) as

$$\Gamma_\mu^{(m)}(p',\ p)=\sum_{i=0}^{\infty}I_i^m[\Delta_F(0)]^{-2i}X_\mu^i(p',\ p), \tag{B·5}$$

where

$$X_\mu^i(p',\ p)=\int_{-\infty}^{\infty}\frac{d^4a}{(2\pi)^4}\int d^4L e^{iLa}[\Delta_F(a)]^{2i}\Gamma_\mu^{(m)(2)}(p'-L,\ p-L). \tag{B·6}$$

We first calculate X_μ^1 to obtain

$$X_\mu^1(p',\ p)=\frac{Z_3}{64(2\pi)^6}\tau_3\int_0^1dx\int_0^1dy\int_0^1\frac{[(1-2y)^2(1-z)-1]dz}{16(1-z)^2x^3(1-y)^3z}\int_0^1u^2du\,J_\mu, \tag{B·7}$$

where

$$J_\mu=\int_0^{\Lambda^2}d\lambda_1\int_0^{\Lambda^2}d\lambda_2\int_0^{\Lambda^2}d\lambda_3\frac{i(b'p'-bp-cp_u)(ip_u+M)}{(\Delta_u-p_u^2)^2}+\frac{1}{2}\gamma_\mu\frac{(\Delta_u-p_u^2)}{(\Delta_u-p_u^2)^2},$$

$$b'=1+\frac{2xz}{(1-2y)^2(1-z)-1},\qquad b=1+\frac{2(1-x)z}{(1-2y)^2(1-z)-1},$$

$$c=2+\frac{2z}{(1-2y)^2(1-z)-1},\qquad p_u=u[(1-x)p+xp'],$$

$$\Delta_u=c_0-c_1\lambda_1-c_2\lambda_2-c_3\lambda_3,$$

$$c_0=\frac{u(M^2B-z\mu^2)}{4y(1-y)z(1-z)}-\frac{u\mu^2}{y(1-y)z}+(1-u)M^2,$$

$$c_1=\frac{u}{y(1-y)z},\qquad c_2=c_3=\frac{u}{A},$$

$$A=\frac{y(1-y)z}{[(1-2y)^2(1-z)-1]^2},$$

$$B=(1-2x+2x^2)\,(1-2y)^2z^2(1-z)$$
$$+[1-z+2xz(1-x)]\,[(1-2y)^2(1-z)-1]. \qquad (B\cdot 8)$$

Intergrations over λ_1, λ_2 and λ_3 may be easily performed to give

$$\int_0^{\Lambda^2} d\lambda_1 \int_0^{\Lambda^2} d\lambda_2 \int_0^{\Lambda^2} d\lambda_3 [\Delta_u - p_u^2]^{-2}$$
$$= \frac{\Lambda^2}{c_2 c_3} \ln \left[\frac{c_0-(c_1+c_3)\Lambda^2-p_u^2}{c_0-(c_1+c_2+c_3)\Lambda^2-p_u^2} \cdot \frac{c_0-(c_1+c_2)\Lambda^2-p_u^2}{c_0-c_1\Lambda^2-p_u^2} \right]$$
$$- \frac{c_0-c_3\Lambda^2-p_u^2}{c_1 c_2 c_3} \ln \frac{c_0-(c_1+c_3)\Lambda^2-p_u^2}{c_0-c_3\Lambda^2-p_u^2}$$
$$+ \frac{c_0-(c_2+c_3)\Lambda^2-p_u^2}{c_1 c_2 c_3} \ln \frac{c_0-(c_1+c_2+c_3)\Lambda^2-p_u^2}{c_0-(c_2+c_3)\Lambda^2-p_u^2}$$
$$- \frac{c_0-c_2\Lambda^2-p_u^2}{c_1 c_2 c_3} \ln \frac{c_0-(c_1+c_2)\Lambda^2-p_u^2}{c_0-c_2\Lambda^2-p_u^2}$$
$$+ \frac{c_0-p_u^2}{c_1 c_2 c_3} \ln \frac{c_0-c_1\Lambda^2-p_u^2}{c_0-p_u^2} \qquad (B\cdot 9)$$

and

$$\int_0^{\Lambda^2} d\lambda_1 \int_0^{\Lambda^2} d\lambda_2 \int_0^{\Lambda^2} d\lambda_3 [\Delta_u - p_u^2]^{-1}$$
$$= -\frac{\Lambda^4}{c_3} \ln \frac{c_0-(c_1+c_2+c_3)\Lambda^2-p_u^2}{c_0-(c_1+c_2)\Lambda^2-p_u^2} + \frac{\Lambda^2[c_0-(c_2+c_3)\Lambda^2-p_u^2]}{c_1 c_3}$$
$$\times \ln \frac{c_0-(c_1+c_2+c_3)\Lambda^2-p_u^2}{c_0-(c_2+c_3)\Lambda^2-p_u^2} - \frac{\Lambda^2[c_0-c_2\Lambda^2-p_u^2]}{c_1 c_3} \ln \frac{c_0-(c_1+c_2)\Lambda^2-p_u^2}{c_0-c_2\Lambda^2-p_u^2}$$
$$+ \frac{\Lambda^2[c_0-(c_1+c_3)\Lambda^2-p_u^2]}{c_2 c_3} \ln \frac{c_0-(c_1+c_2+c_3)\Lambda^2-p_u^2}{c_0-(c_1+c_3)\Lambda^2-p_u^2}$$
$$- \frac{c_0-(c_2+c_3)\Lambda^2-p_u^2}{2c_1 c_2 c_3} (c_0-p_u^2) \ln \frac{c_0-(c_1+c_2+c_3)\Lambda^2-p_u^2}{c_0-(c_2+c_3)\Lambda^2-p_u^2}$$
$$+ \frac{(c_0-c_3\Lambda^2-p_u^2)^2}{2c_1 c_2 c_3} \ln \frac{c_0-(c_1+c_3)\Lambda^2-p_u^2}{c_0-c_3\Lambda^2-p_u^2}$$
$$- \frac{\Lambda^2(c_0-c_1\Lambda^2-p_u^2)}{c_2 c_3} \ln \frac{c_0-(c_1+c_2)\Lambda^2-p_u^2}{c_0-c_1\Lambda^2-p_u^2}$$
$$+ \frac{c_0-c_2\Lambda^2-p_u^2}{2c_1 c_2 c_3} (c_0+c_2\Lambda^2-p_u^2) \ln \frac{c_0-(c_1+c_2)\Lambda^2-p_u^2}{c_0-c_2\Lambda^2-p_u^2}$$
$$- \frac{(c_0-p_u^2)^2}{2c_1 c_2 c_3} \ln \frac{c_0-c_1\Lambda^2-p_u^2}{c_0-p_u^2}. \qquad (B\cdot 10)$$

Similarly one may write $\Delta_F(0)$ as a function of cut-off momentum, Λ^2, as follows:

$$\Delta_F(0)=\frac{-2i}{(2\pi)^4}\int\frac{d^4k}{k^2+\mu^2}=\frac{-4i}{(2\pi)^4}\int_0^{\Lambda^2}d\lambda_1\int_0^{\Lambda^2}d\lambda_2\int\frac{d^4k}{(k^2+\mu^2+\lambda_1+\lambda_2)^3}$$

$$=\frac{2\pi^2}{(2\pi)^4}\left[(2\Lambda^2+\mu^2)\ln\frac{2\Lambda^2+\mu^2}{\Lambda^2+\mu^2}-\mu^2\ln\frac{\Lambda^2+\mu^2}{\mu^2}\right]. \qquad \text{(B·11)}$$

Writing

$$\bar{u}(p')X_\mu^i(p'\ p)u(p)=\bar{u}(p')\left[\gamma_\mu x^i(k^2)+\frac{\sigma_{\mu\nu}k_\nu}{2M}\gamma^i(k^2)\right]u(p), \qquad \text{(B·12)}$$

we see from eqs. (B·7)$\sim$(B·10) that the worst singularities of $x^1(k^2)$, $\gamma^1(k^2)$ and $[\Delta_F(0)]^2$ are

$$x^1(k^2)\sim\Lambda^4\ln(\Lambda^2/M^2),\qquad \gamma^1(k^2)\sim\Lambda^2\ln(\Lambda^2/M^2),$$

$$[\Delta_F(0)]^2\sim\Lambda^4.$$

Accordingly, in the limit of $\Lambda\to\infty$, contributions of X_μ^1 to $\Gamma_\mu^{(m)}(p',\ p)$ vanishes except $x^1(0)/[\Delta_F(0)]^2$. Similar situation holds also for the case of $X_\mu^i(i\geqq 1)$ as will be seen from the form of eq. (B·6).

So, if we write

$$\bar{u}(p')\Gamma_\mu^{(m)}(p',\ p)u(p)=\sum_{i=0}^{\infty}I_i^m[\Delta_F(0)]^{-2i}\bar{u}(p')X_\mu^i(p',\ p)u(p)$$

$$\equiv\bar{u}(p')\left[\gamma_\mu F_1^{(m)}(k^2)+\frac{\sigma_{\mu\nu}k_\nu}{2M}F_2^{(m)}(k^2)\right]u(p), \qquad \text{(B·13)}$$

$X_\mu^i(i\geqq 1)$ contribute only to $F_1^{(m)}(0)$. Therefore we need only to evaluate the zeroth order term in I^m to obtain observable quantities.

Above arguments hold also for the evaluation of I_1^n, $\cdots$, I_6^n.

Appendix C

In eq. (4·2) $\Delta_F(x-y)$ appear in the exponential form, and one might think that this represents the propagation of many pions from x to y. That this is not the case may be seen from the results obtained in Appendix B.

In this Appendix, we intend to investigate the above-mentioned situation more in detail and make some comments on the renormalization procedure proposed by several authors[12)13)] to the case of usually unrenormalizable theories.

Consider the neutral scalar meson field coupled with the two static sources at $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, the interaction Hamiltonian being given by

$$H_1(x)=\lambda\varphi(x)[\delta(\boldsymbol{x}-\boldsymbol{x}_1)+\delta(\boldsymbol{x}-\boldsymbol{x}_2)]. \qquad \text{(C·1)}$$

Then nuclear potential is easily obtained and the result coincides exactly with the lowest order perturbation result.

Next we consider the case, where the interaction Hamiltonian is given by

$$H_2(x) = \lambda \int du\, e^{-i\varphi(x)u} f(u) [\delta(\boldsymbol{x}-\boldsymbol{x}_1) + \delta(\boldsymbol{x}-\boldsymbol{x}_2)], \tag{C·2}$$

to compare the situation with eq. (4·2). Nuclear potential in this case is given by

$$V(x-x') = -\tfrac{1}{2} i \lambda^2 \iint du\, du' f(u) f(u') \langle [e^{-i\varphi(x)u},\ e^{-i\varphi(x')u'}] \rangle_0 \epsilon(x-x')$$

$$= \lambda^2 \iint du\, du' f(u) f(u') \exp\left[-\tfrac{1}{4}\{u^2 \Delta^{(1)}(0) + u'^2 \Delta^{(1)}(0)\}\right]$$

$$\times \mathrm{Im}\left[\exp\{-\tfrac{1}{2} u u' \Delta_F(x-x')\}\right], \tag{C·3}$$

up to the second order in λ. Eq. (C·3) is very similar to eq. (4·2), and indeed represents the exchange between two sources of may mesons in general. However, if one put

$$f(u) = (2\pi)^{-1} \int d\chi \exp(i\chi u), \tag{C·4}$$

then $H_2(x)$ is equal to $H_1(x)$ and the number of mesons propagating from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ should be one. Inserting (C·4) into eq. (C·3) and integrating over u and χ, one obtains

$$V(x-x') = -\lambda^2 \bar{\Delta}(x-x'), \tag{C·5}$$

which coincides exactly with the nuclear potential derived from $H_1(x)$, as it should be. One may say about the number of mesons propagating from x to y only after performing the integrations over parameters. The situation is exactly the same for eq. (4·2), the number of propagating mesons being same with the second order perturbation calculation.

Glauber[12] has shown for the coupling

$$H(x) = \lambda \rho(\boldsymbol{x}) e^{\beta\varphi(x)}, \tag{C·6}$$

which is obtained from eq. (C·2) by replacing $f(u) = \delta(u - i\beta)$ symbolically and replacing $\delta(\boldsymbol{x}-\boldsymbol{x}_1) + \delta(\boldsymbol{x}-\boldsymbol{x}_2)$ by $\rho(\boldsymbol{x})$, that S-matrix contains no infinities after the renormalization

$$\lambda' = \lambda \exp\left[\tfrac{1}{4}\beta^2 \Delta^{(1)}(0)\right]. \tag{C·7}$$

This type of renormalization has also been applied to the case of usually unrenormalizable theories by Arnowitt and Deser, by Cooper and by Green.[13] If one intends to follow their prescriptions, one would renormalize λ^2 in eq. (C·3) by

$$\lambda'^2 = \lambda^2 \exp\left[-\tfrac{1}{4}\{u^2 \Delta^{(1)}(0) + u'^2 \Delta^{(1)}(0)\}\right]. \tag{C·8}$$

However, if one adopt $f(u)$ given by (C·4), the integrations in the right side of eq. (C·3) may be easily performed to give eq. (C·5), where apparently no renormalization is needed. So we may not renormalize λ^2 by (C·8) in our case.

Moreover, if $\exp\left[-\tfrac{1}{4}\{u^2 \Delta^{(1)}(0) + u'^2 \Delta^{(1)}(0)\}\right]$ is absorbed into λ'^2, the absolute convergency of the integrals in eq. (C·3), which does hold originally, ceases to hold, and one encounters some difficulties in the integrations.

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Letters to the Editor

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On a Non-Adiabatic Approximate Method in the Non-Linear Interaction Theory

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It has been shown by many authors that the perturbation method in quantum field theory leads to divergent series. B. L. Ioffe[2] showed that e^2-powers expansions of the total Lagrangian in a uniform electric field diverge in the special case of quantum electrodynamics, which suggests the divergency even in the general case of Q. E. D. Such a divergence of power series expansion is due to the non-linearity of the interaction Hamiltonian which results from the requirement of the conservation law of momentum. For the sake of simplicity, we consider the scalar field interacting with the self-field. In order to separate the divergency due to the power series expansion and the divergency due to the infinite degrees of freedom of fields, we will consider without loss of generality an anharmonic oscillater with the interaction term λq^4, where λ is a coupling constant. Even in the case of an anharmonic oscillater, the power series expansion leads to divergence, whose proof will be shown later. Therefore it is necessary to seek for a non-adiabatic method. Here we develope a non-adiabatic procedure by Feynman's functional integral method. The kernel for an anharmonic oscillater is given by the following:

$$K(q', q'', t', t'') = \int\cdots\int \exp\left[\frac{i}{2}\int_{t'}^{t''} (\dot{q}^2-\omega^2 q^2-2\lambda q^4)\,dt\right] \mathcal{D}q. \tag{1}$$

The term which contains q^4 is expressed as follows by using the functional Fourier's transformation:

$$\exp\left[-i\lambda\int q^4 dt\right] = \int\cdots\int \exp\left[i\lambda\int (x^2-2xq^2)\,dt\right] \mathcal{D}x. \tag{2}$$

Then

$$K(q', q'', t', t'') = \int\cdots\int \exp\left[\frac{i}{2}\int (\dot{q}^2-\omega^2 q^2-4\lambda x q^2)\,dt\right] \mathcal{D}q \times \exp\left[i\lambda\int x^2 dt\right] \mathcal{D}x, \tag{3}$$

which means the weighted superposition of harmonic oscillators with a time-dependent frequency. The functional integrations with respect to q are exactly performed, which will be shown soon later;

$$K(q', q'', t', t'') = \int\cdots\int \frac{1}{\sqrt{2\pi i y_2(t'')}} \exp\left[\frac{i}{2y_2(t'')}\{\dot{y}_2(t'')q''^2$$

$$+\gamma_1(t'')q''^2-2q'q''\}\Big]\exp[i\lambda\int x^2dt]\,\mathfrak{D}x, \tag{4}$$

where $\gamma_1(t)$ and $\gamma_2(t)$ are the special solutions of the differential equation $\ddot{y}+(\omega^2+4\lambda x)y=0$ and satisfy the following initial conditions: $\gamma_2(t')=0$, $\dot{\gamma}_2(t')=1$, $\gamma_1(t')=1$, $\dot{\gamma}_1(t')=0$. Hereafter we put $\omega=0$ for the sake of simplicity. In order to expand the formula (3) as the series of products of Hermite polynomials, we put $\gamma_1(t)$ and $\gamma_2(t)$ as follows:

$$\gamma_1(t)=f(t')\cos\int_{t'}^{t}f^2(\tau)d\tau/f(t),$$

$$\gamma_2(t)=\gamma_1(t)\int_{t'}^{t}d\tau/\gamma_1(\tau)^2$$

$$=\sin\int_{t'}^{t}f^2(\tau)d\tau/f(t')f(t), \tag{5}$$

where $f(t)$ satisfies the differential equation

$$f^4+\ddot{f}f-2\dot{f}^2/f^2=4\lambda x \tag{6}$$

and the initial condition $\dot{f}(t')=0$. Now we transform the variables $x(t)$ to $f(t)$ in order to perform the functional integration of the formula (4). Then

$$K(q',\,q'',\,t',\,t'')$$

$$=\int\cdots\int\sqrt{f(t')f(t'')}\;2\pi i\sin\int_{t'}^{t''}f^2dt$$

$$\times\exp\Big[\frac{i}{2}\cot\int_{t'}^{t''}f^2dt(f^2(t')q'^2+f^2(t'')q''^2)$$

$$-if(t')f(t'')q'q''\operatorname{cosec}\int_{t'}^{t''}f^2dt$$

$$-\frac{i}{2}\frac{\dot{f}(t'')}{f(t'')}q''^2\Big]$$

$$\times\exp\Big[\frac{i}{16\lambda}\int_{t'}^{t''}\{f^4+\ddot{f}f-2\dot{f}^2/f^2\}^2dt\Big]$$

$$\times\exp\Big[\frac{i}{\lambda}\int_{t'}^{t''}f^4dt\Big]\Pi\frac{-1}{4\lambda\varepsilon^2f(t_i)^{1+2\varepsilon}}df(t_i). \tag{7}$$

If we apply the generating function of bilinear Hermite polynomials to (7),

$$\frac{1}{\sqrt{1-t^2}}\exp\Big\{-\frac{t^2(x^2+y^2)-2txy}{2(1-t^2)}\Big\}$$

$$=\sum_n\frac{t^n}{n!}H_n(x)H_n(y),$$

then we get

$$K=\sum_n\frac{1}{2^n n!}\int\cdots\int\sqrt{\frac{f(t')f(t'')}{\pi}}$$

$$\times\exp\Big[-\frac{i}{2}\frac{\dot{f}(t'')}{f(t'')}q''^2\Big]$$

$$\times\exp\Big[-i\Big(n+\frac{1}{2}\Big)\int_{t'}^{t''}f^2dt\Big]$$

$$\times\exp\Big[-\frac{1}{2}\{f(t')^2q'^2+f(t'')^2q''^2\}\Big]$$

$$\times H_n(f(t')q')H_n(f(t'')q'')$$

$$\times\exp\Big[\frac{i}{16\lambda}\int_{t'}^{t''}\{f^4+\ddot{f}f-2\dot{f}^2/f^2\}^2dt\Big]$$

$$\times\exp\Big[\frac{i}{\lambda}\int_{t'}^{t''}f^4dt\Big]\Pi\frac{-1}{4\lambda\varepsilon^2f(t_i)^{1+2\varepsilon}}df(t_i). \tag{8}$$

It should be noted that as (7) and (8) contain the second derivative with respect to t, the multiplicative law of a kernel does not hold in this approximation. As the most contribution of the functional integral (8) comes from the part f^4, we take only the part of f^4 as the zeroth approximation. As f which gives the largest contribution to the integral is

$$f^2=\sqrt[3]{\lambda(2n+1)\pm\sqrt{\lambda^2(2n+1)^2-(8/3)^3}}$$

by using the saddle point method, the energy eigenvalues are expanded as $\lambda^{-2/3}$ for $\lambda \gg 1$:

$$E_1 = -(2n+1/4)^{4/3}\lambda^{1/3} + 2^{2/}(2n+1)^{2/3}\lambda^{-1/3}+\cdots\cdots$$
$$E_2 = -8/3\times 2^{2/3}\lambda^{-1/3}(2n+1)^{2/3}+\cdots\cdots \qquad (9)$$

and the eigenfunction in this zeroth approximation is given by

$$\psi_n(q') = \left(\frac{2n+1}{4}\right)^{1/6}\lambda^{1/6}\left(\frac{2}{\pi}\right)^{1/4}\frac{1}{\sqrt{n!}} \times \exp\left[-\left(\frac{2n+1}{4}\lambda\right)^{1/3}q'^2\right] \times H_n\left(\sqrt{2}\left[\frac{2n+1}{4}\lambda\right]^{1/6}q'\right). \qquad (10)$$

If we tend $n\to\infty$, $\lambda\to 0$ by fixing the energy $E=(2n+1/4)^{4/3}\lambda^{1/3}$ in this zeroth approximate eigenfunction, $\psi_n(q')$ tends to

$$\psi_E(q') \sim \lambda^{3/16}\exp[E^{1/4}q\lambda^{-1/4} \pm i\sqrt{2E}q'],$$

whose normalization constant contains an essential singular point of λ and does not tend to the usual normalized plane wave. This means the convergency of this approximate method, whose proof will be shown later.

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An Example of One-Dimensional Oscillatory Motion in Magnetohydrodynamics

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It is the aim of this note to analyse an example of one-dimensional oscillations of a highly conductive medium caused by some external force. Such a problem has already been treated by Kulikovsky in connection with the pulsation of a cylindrical plasma in the hign current discharge.[1] Employing a procedure analogous to his method, we can find a set of particular solutions of the basic equations in one dimension. Then we get a peculiar feature of magnetohydrodynamics that the moving region of the medium can be adjacent to the motionless part. This affords us, at the same time, an example which does not conform to the fundamental theorem on the simple wave zone in the flow problem.

The one-dimensional motion of the highly conductive medium is described by the magnetohydrodynamic equations in Lagrange's form

$$\frac{\partial^2 x}{\partial t^2} = -\frac{1}{\rho}\left\{\frac{\partial p}{\partial x}+\frac{1}{8\pi}\frac{\partial H^2}{\partial x}\right\}, \qquad (1)$$

$$\rho\, dx = \text{const}, \quad H dx = \text{const}, \quad p\rho^{-\gamma} = \text{const}, \qquad (2)$$

where x is the particle coordinate, H the magnetic field which is assumed to be perpendicular to the direction of motion of the particle. The last three equations represent the conservation of mass, magnetic flux, and entropy, respectively, along the

path of the particle.

We seek a particular solution of the equations by assuming

$$x=x_0+(a-x_0)\hat{\xi}(t),$$

$$\text{for } 0\leq x_0\leq a \text{ (region I)}, \tag{3a}$$

$$x=x_0, \text{ for } a\leq x_0 \text{ (region II)}. \tag{3b}$$

For the boundary $x_0=0$ we specify later. Here and in what follows the quantities with suffix 0 pertain to the initial instant of time. In (3a), $\hat{\xi}(t)$ is independent of x_0, $\hat{\xi}(0)=0$ and, as will be seen below, $\hat{\xi}(t)<1$, $\hat{\xi}'(0)=0$.

In the region I, the conservation laws become

$$\rho=\frac{\rho_0}{\eta},\quad H=\frac{H_0}{\eta},\quad p=\frac{p_0}{\eta^\gamma}, \tag{2'}$$

with $\eta(t)=1-\hat{\xi}(t)$. Eq. (1) then reduces to

$$(\eta')^2=\frac{A}{\gamma-1}\frac{1}{\eta^{\gamma-1}}+\frac{B}{\eta}+C, \tag{4}$$

in which

$$A=-\frac{2}{\rho_0(a-x_0)}\frac{\partial p_0}{\partial x_0}, \tag{5}$$

$$B=-\frac{1}{4\pi\rho_0(a-x^0)}\frac{\partial H_0^2}{\partial x_0}, \tag{6}$$

and C are constants independent of x_0, because the left-hand side of (4) is the function of t alone. Conversely an appropriate choice of three constants A, B, and C not only characterizes the motion of the medium from (4), but also determines the initial distribution of pressure and magnetic field in terms of the density by integrating (5) and (6).

Since $1<\gamma\leq 5/3$ for usual media, eq. (4) has the solution only for positive $\eta(t)$ (or $\hat{\xi}(t)<1$), as expected. For the periodic solution, in which we are mainly interested, we have a set of conditions,

$$A>0,\ B<0 \text{ and } C<0.$$

As to other possible choices of constants, it suffices to note that they lead ultimately to either the compression or the expansion of the medium without accompanying any discontinuity such as a shock.

It is clear from (5) and (6) that if $A>-B$, the pressure gradient exceeds the magnetic pressure gradient at $t=0$ and the periodic motion starts with the compression of the medium. If $A<-B$, on the other hand, the initial acceleration is negative, showing that the expansive motion occurs at first.

In the region II, where the medium is at rest, the total pressure must be spatially constant. The only requirement at the boundary $x_0=a$ is the balance of the total pressure. No particle crosses the boundary and the particle velocity is the same on its both sides. These features are similar to those of the contact discontinuity[2]; but here the density, the pressure and the magnetic field may change continuously across the boundary. Further it is worthwhile to remark that the region I is not of simple wave, while its adjacent region II is of constant state. Thus, due to the presence of magnetic field, we have an exception of the fundamental theorem in flow problems that the flow in a region adjacent to a region of constant state is a simple wave.[2),3)] We have also an interesting case where there is no medium in the region II and the total pressure is supplied by an external magnetic field alone. On these points, we will discuss in more detail elsewhere.

Whatever initial distribution of density

ρ_0 one may choose, we cannot make the total pressure at the boundary $x_0=0$ equal to zero, because $p-(A/B)\eta^{-(\gamma-2)}(H^2/8\pi)$ is constant for $a\xi<x<a$, as seen from (2′), (5), and (6). That is, if the total pressure is zero at the boundary $x=a\xi$, then it vanishes everywhere in this interval. It might be suggested that by putting $x=x_0+(a+x_0)\tilde{\xi}(t)$, for $-a<x_0<0$, we could balance the total pressure at $x=a\xi$ and obtain an oscillation of a limited region of the medium $(-a<x<a)$. However, this is impossible, since $\tilde{\xi}(t)$ has no necessary symmetry property. The motion given by (3a) seems to need, therefore, at the boundary $x_0=0$, some external force like a piston or an applied magnetic field.

In the isentropic case for which $p_l\rho^{-\gamma}=\alpha$ is a constant independent of x_0, the initial density distribution in the region I is not arbitrary but is determined by the equation

$$\rho_0^{\gamma-2}\frac{\partial\rho_0}{\partial x_0}=-\frac{A}{2\alpha\gamma}(a-x_0).$$

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Absorption Effects in Antinucleon Phenomena

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It was shown experimentally[1] that the ratio of the antiproton production cross section for a nucleon in a carbon nucleus to that of a free proton is 0.77, and it was expected that the reabsorption of antiproton produced in a nucleus should be taken into account. On the other hand, the cross sections of antineutron production by charge exchange of antiproton for the cases of a free proton, carbon and lead nuclei were determined as 3, 4 and 3.8 mb, respectively.[2] The experimental fact that the antineutron production cross section is independent of mass number of proton number of the target nucleus, may possibly be interpreted by absorption effects of antinucleons in nuclear matter.

The absorption effects in antinucleon phenomena will be examined in the present note in order to explain the above mentioned experiments. Since the antinucleon production cross sections for elementary processes are known in both cases, only the consideration for the process of absorption in a nucleus is necessary.

(I) *Antiproton production by 6.1 Gev proton beam.* In the first place, the statistical calculation including only the effect of the intranuclear motion of nucleons indicates that the antiproton production cross section per nucleon σ' in carbon is

about 8.5 times as large as that of hydrogen $\sigma(H)$. This result shows the discrepancy of about one-order in comparison with the experimental one. Hence one must take the absorption and reabsorption effects of incident proton beam and produced antiprotons. The antiproton production cross section (per nucleon) in which the absorption effects in the nuclear matter are taken into account based on a very simple model is expressed as

$$\sigma=\frac{2\pi}{A}\int_0^R bdb\int_0^{2S} ds_1\,\sigma'\rho \times\exp[-s_1/\lambda_\pi-(2S-s_1)/\lambda_\alpha],$$

where λ_π and λ_α are the mean free paths of proton for the multiple π-production process and of antiproton for the annihilation process, respectively, and ρ is the nucleon density in the nucleus. In this expression, the effects of transmission and refraction at the nuclear surface are neglected, and it is assumed that the antinucleon phenomena occur on a line along the direction of the incident particle.

In the evaluation of this expression, the nuclear radius R is chosen as $1.3\times A^{1/3}\times 10^{-13}$ cm and for the mean free paths, $\lambda_\pi=2.9\times10^{-13}$ cm and $\lambda_\alpha=1.1\times10^{-13}$ cm (for a carbon nucleus) are used conveniently. Inserting these values into the above expression, one finds σ (for carbon) $=0.10\times\sigma'$ (for carbon) and for the ratio of the antiproton production cross section for a nucleon in a carbon nucleus to that of a free proton, σ (for carbon)$/\sigma(H)\simeq0.88$, which has to be compared with the experimental ratio, 0.77. The result seems to be in good agreement with the experimental one since the latter is not so accurate.

(II) *Antineutron production by charge exchange of 440 Mev antiproton.* In this case the effects of absorption are treated as in the case of antiproton production. For the heavy nucleus, two different nuclear models are taken: first as model-A, we assume that R_p and R_n, the radius of proton ond neutron distribution, respectively, to be the same. Secondly, making use of the indication that the protons are confined to a region of a somewhat smaller radius compared with the nucleus as a whole due to the Coulomb potential, we assume $R_n>R_p$, as model-B. Since the antineutrons may be produced only by antiproton-proton collisions at this energy, the antineutron production cross section (not per nucleon but for nucleus) is written by

$$\sigma=4\pi\sigma_{ex}\rho_p\int_0^{R_p}(S-s_n)\times\exp[-2(S-s_n)/\lambda_i^a-2s_n/\lambda_0^a]\,bdb,$$

where σ_{ex} is the elementary cross section for charge process, ρ_p is the proton density, s_n is the width of the outer neutron shell along the particle path at the distance of b from the center of the nucleus, and λ_i^a and λ_0^a are the annihilation mean free paths of antinucleon in the inner region with radius R_p and those in the outer neutron shell region, respectively. We estimate this expression for each case below.

i) For the case of a carbon nucleus. It may be chosen as $R_n=R_p=R$ (nuclear radius) and $\lambda_i^a=1.1\times10^{-13}$ cm, then we find $\sigma=0.22\times\sigma_{ex}$.

ii) For the case of a lead nucleus-model-A. We may choose $R_n=R_p=R$ and $\lambda_i^a=1.2\times10^{-13}$ cm, and so this results in $\sigma=0.20\times\sigma_{ex}$.

iii) For the case of a lead nucleus-model-B. The choice of parameters in this case is as follows, $R_n = R = 1.3 \times A^{1/3} \times 10^{-13}$ cm, $R_p = 1.18 \times A^{1/3} \times 10^{-13}$ cm, $\lambda_0^{\alpha} = 2.1 \times 10^{-13}$ cm and $\lambda_i^{\alpha} = 1.0 \times 10^{-13}$ cm. By the use of these values, one obtains $\sigma = 0.023 \times \sigma_{ex}$.

According to the above calculations, model-A for lead seems to be consistent as to mass number (or proton number) dependence, but it gives about 1/6 of the experimental cross section just as in the case of a carbon nucleus. On the other hand, model-B seems to be excluded since the calculated value is smaller than the one for carbon by one order of magnitude. However, one must consider other effects of nuclear matter on the antineutron production mechanism such as the diffuseness of the nuclear surface or other intranuclear effects. As another important fact, it may be noted that the very large interaction cross sections of antiprotons on complex nuclei are reported experimentally.

The authors would like to express their gratitude to Professor K. Husimi for his guidance. They are also indebted to Messrs. S. Yamasaki, Y. P. Kwon and K. Kobayakawa for many helpful discussions.

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Equivalent Hamiltonian of an Electron Gas at High Density

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Recently Wentzel[1] has presented a method of the so-called equivalent Hamiltonian of an electron gas based upon Sawada's theory.[2] In this theory the Bloch interaction has been adopted as the interaction between electrons and phonons together with the mutual Coulomb interaction of electrons in the explicit form. From our point of view, if the Bloch interaction is assumed, the mutual interaction of electrons $V^{(e)}$ should be of a screened form. Alternatively, if the mutual interaction $V^{(e)}$ is assumed in the explicit Coulombic form we should adopt the attractive Coulombic potential[3] as the interaction between electrons and ions in place of the Bloch interaction. Therefore some modification becomes necessary to derive physically meaningful results. Wentzel's equivalent Hamiltonian is very similar to our Hamiltonian of coupled oscillators proposed in previous papers.[4] Both for an electron gas and for an electron-ion plasma it can be shown that results derived from the one are in fair agreement with those from the other.

We consider that the ionic motion is represented by a harmonic oscillation with the ionic plasma frequency $\Omega_0 = (4\pi e^2 z^2 N/M)^{1/2}$, where N, z, and M are the ionic density, charge, and mass. Then Wentzel's dispersion equation is changed to[4]

$$\phi_q(\nu^2) = 1 + \frac{2\lambda_q \nu^2}{\Omega_0^2 - \nu^2} \sum_F \frac{\omega_{p\,q}}{\nu^2 - \omega_{p-q}^2},$$

$$\omega_{p,q} = (\boldsymbol{p}\cdot\boldsymbol{q}), \qquad (1)$$

$$m = \hbar = 1,$$

setting $\nu_q^2 = \Omega_0^2$ and $\lambda_q = |\mu_q|^2/2\Omega_0^2 = 4\pi e^2/q^2$ in Wentzel's result (7a). In the continuum limit the phonon frequency given by this dispersion equation becomes imaginary.[1)]

Next, to gain a great familiarity with the current theories and to investigate the damping of phonons, we should like to diagonalize the Hamiltonian of electrons in the absence of the ionic oscillation and to see to what extent the frequency of the ionic oscillation is shifted by eliminating the interaction between the electronic plasma oscillation and the ionic oscillation. Thus, after separating the plasma mode from the Hamiltonian we consider the interaction between individual electrons and phonons, quanta of the ionic oscillation modified by the plasma oscillation. This interaction resembles Froelich's interaction[5)] familiar in the theories of superconductivity. The coupling parameter corresponding to Froelich's becomes $M(q) = \pi(\Omega_s/P_0)^{1/2}$, $z=1$ and $2(\Omega_s/P_0)^{1/2}$ for the momentum transfers of individual electrons parallel and perpendicular to the Fermi surface, respectively, where P_0 stands for the Fermi momentum and $\Omega_s = (3z/5M)^{1/2} P_0 q$ implies the phonon frequency.

Finally, we should like to give the change in Gell-Mann's correlation energy caused by the ionic oscillation.[6)] It can be obtained from the dispersion equation by performing a contour integral enclosing the zero points and the poles of $\phi_q(\nu^2)$. Taking the contour along the imaginary axis we get

$$\Delta\epsilon_{\mathrm{corr.}}^{(G)} = (3/4)\int_0^1 q^2 dq \int_{-\infty}^{\infty} (du/2\pi)$$

$$\times \ln[1 - \{X_q(u)\, Y_q(u)/(1+X_q(u))\}], \qquad (2)$$

$$X_q(u) = (4e^2/\pi P_0 q^2)\, R(u/P_0^2 q),$$

$$R(x) = 1 - x\tan^{-1}x^{-1}, \qquad (3)$$

$$Y_q(u) = \Omega_0^2/(\Omega_0^2 + u^2). \qquad (4)$$

This result has been confirmed by a field theoretical perturbation method similar to Gell-Mann's theory for our coupled oscillator model. A fuller presentation will be given elsewhere in the near future. The author thanks Professor N. Fukuda for his valuable comments and discussions.

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Coupling Constant of Kaon-Baryon Interaction, I

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Although there have been several attempts

to investigate the dynamical properties of kaons and baryons, even their coupling type and coupling strength are yet unknown. In this note, assuming a simple coupling type, we shall point out that a physical requirement on the low energy $K-N$ scattering restricts the magnitude of the coupling constant.

We assume that the spins of baryons are all $1/2$ and that of kaon is 0, Λ and Σ are of the same parity and $(N\Lambda K)$-, $(N\Sigma K)$- interactions are of the scalar type. All effects of pions and anti-kaons are taken into account, but direct interactions of kaons with pions are not considered. Despite the large kaon mass, recoil effects are found to be fairly small in the low energy $K-N$ reactions through the scalar interaction and do not alter the results of this note more than 20%. In the following we use the static theory[1] from the beginning.

The T-matrix element for the reaction $K^+ + p \to K^+ + p$ is written as $-2\pi(\omega_q \omega_p)^{-1/2} h(\omega_p)$. Here the indexes q and p indicate the initial and final kaon momenta and $\omega_p = (m_k^2 + p^2)^{1/2}$. The elastic cross section is $\sigma_{el} = 4\pi |h(\omega_p)|^2$ and the total cross section is $\sigma_{tot} = 4\pi \operatorname{Im} h(\omega_p)/p$. The total cross section for the $K^- + p$ reaction is denoted by $\tilde{\sigma}_{tot}$. The Low equation for $h(\omega_p)$ is

$$h(\omega_p) = \frac{\lambda}{\omega_p + \Delta m} + \frac{1}{4\pi^2} \int_{m_K} d\omega_q\, q \times \left(\frac{\sigma_{tot}(\omega_q)}{\omega_q - \omega_p - i\epsilon} + \frac{\tilde{\sigma}_{tot}(\omega_q)}{\omega_q + \omega_p} \right) + \frac{1}{4\pi^2} \int_{m_\pi + \Delta m}^{m_K} d\omega_q\, q \frac{\tilde{\sigma}'_{cap}(\omega_q)}{\omega_q + \omega_p}. \quad (1)$$

Here $\lambda = g_\Lambda^2 + g_\Sigma^2$, g_Λ and g_Σ being the renormalized coupling constants of $(N\Lambda K)$- and $(N\Sigma K)$-interactions, and $\Delta m\,(\approx 0.5\, m_K)$ is the mass difference between hyperon and nucleon; mass difference between Λ and Σ is neglected. $\tilde{\sigma}'_{cap}(\omega_p)$ $(\omega_p < m_K)$ is a formal extension of the K^- capture cross section $\tilde{\sigma}_{cap}$ into the unphysical region.

With the presumption that $h(\omega_p)$ has no zeros on the real axis of the complex ω-plane, we have[1),2)]

$$\operatorname{Re} h(\omega_p)^{-1} = \lambda^{-1} \cdot (\omega_p + \Delta m)\,\{1 - \lambda \cdot A(\omega_p)\}, \quad (2)$$

$$A(\omega_p) = \frac{\omega_p + \Delta m}{\pi} \Bigg[\int_{m_K} d\omega_q \left\{ f_{+-}(\omega_q, \omega_p) \times \frac{\sigma_{tot}(\omega_q)}{\sigma_{el}(\omega_q)} + f_{-+}(\omega_q, \omega_p) \frac{\tilde{\sigma}_{tot}(\omega_q)}{\tilde{\sigma}_{el}(\omega_q)} \right\} + \int_{m_\pi + \Delta m}^{m_K} d\omega_q f_{-+}(\omega_q, \omega_p) \frac{\tilde{\sigma}'_{cap}(\omega_q)}{\tilde{\sigma}_{el}(\omega_q)} \Bigg], \quad (3)$$

where

$$f_{\pm\mp}(\omega_q, \omega_p) = \mathrm{P} \cdot q / (\omega_q \pm \Delta m)^2 (\omega_q \mp \omega_p).$$

P stands for the principal value. $\tilde{\sigma}_{el}$ is the cross section for the pure elastic reaction $K^- + p \to K^- + p$. The low energy scattering under present consideration is elastic $(\sigma_{tot} = \sigma_{el})$ and the real phase shift is given by $p \cot \delta(p) = \operatorname{Re} h(\omega_p)^{-1}$.

Each term of eq. (3) is decreasing for $\omega_p > m_K$, and positive in the low energy region. Then, if the value of λ is as large as $\lambda \cdot A(m_K) = 1$, resonounce occurs at $\omega_p = m_K$. For a larger value than this, $h(\omega)$ acquires a new pole at $\omega < m_K$ and correspondingly an inhomogeneous term should be added to the Low eq. (1).[3] Then the phase shift starts from 180°. This implies that

a bound state of K and N comes into existence. Whereas such a particle with strangeness $+1$ and mass less than M_N+m_K has never been observed. Therefore we may put a restriction on λ;

$$\lambda<\hat{\lambda}=A(m_K)^{-1}. \tag{4}$$

Now we estimate the upper bound $\hat{G}$. Though $\sigma_{tot}/\sigma_{el}>1$ at high energies we substitute unity for this ratio in the integrand of eq. (3). Write $\int_{m_K}^{\omega_{\max}} d\omega_q f_{\pm\mp}(\omega_q, \omega_p) \equiv F_{\pm\mp}(\omega_p)$ and introduce a function $R(\omega_p)$ by

$$\int_{m_K}^{\omega_{\max}} d\omega_q f_{-+}(\omega_q, \omega_p)\,\tilde{\sigma}_{tot}/\tilde{\sigma}_{el} \equiv R(\omega_p)\,F_{-+}(\omega_p). \tag{5}$$

Then

$$\hat{\lambda}=\pi[(m_K+\Delta m)(F_{--}+RF_{-+}+C)_{\omega_p=m_K}]^{-1}. \tag{6}$$

$F_{\pm\mp}(\omega_p)$ and consequently $\hat{\lambda}$ depend on the cutoff energy $\omega_{\max}$. C is the contribution from the unphysical region. As for the order of magnitude of C, a crude estimate will be given as follows. The cross section for K^- capture is expected to obey the $1/v$-law; we assume that

$$\tilde{\sigma}'_{cap}/\tilde{\sigma}_{el}=\text{const}\cdot p_\pi{}^3/m_\pi{}^2 q \tag{7}$$

with $p_\pi=\sqrt{(\omega_q-\Delta m)^2-m_\pi{}^2}$. The factor $p_\pi{}^3$ comes from that the pion is produced in p-wave state. Notice that the denominator q of eq. (7) cancels the factor q in $f_{-+}(\omega_q, \omega_p)$. The experiment[1] shows that $\tilde{\sigma}_{cap}/\tilde{\sigma}_{el}\lesssim 0.3$ for the kinetic energy $10\sim 50$ Mev in the center-of-mass system. Eq. (7) with the const. $\lesssim 0.1$ roughly fits the experiment. Then eq. (7) yields $C(m_K)\lesssim F_{+-}(m_K)/30$, which is negligible as expected from the smallness of the integration interval.

As for $R(m_K)$ the main contribution to the integral of RF_{-+} comes from the low energy region where $\tilde{\sigma}_{cap}/\tilde{\sigma}_{el}\lesssim 0.3$. The cross section for the charge exchang scattering $K^-+p\to\tilde{K}^0+n$ is not well known, but is likely smaller than σ_{el}.[4] Therefore $R(m_K)=1.5\sim 2.0$ would be an appropriate assumption.

Taking $R(m_K)=1$ and 2, and dropping C, $\hat{\lambda}$ is plotted in Fig. 1 versus $\omega_{\max}=2\sim 5\,m_K$. If, for instance $R(m_K)\approx 2$ and $\omega_{\max}=2.5\,m_K$ ($\approx$hyperon mass) is assumed, we have $\hat{\lambda}\approx 1.0$. A more accurate estimate, taking account of $\sigma_{tot}/\sigma_{el}>1$ and $C>0$, gives more severe restriction.

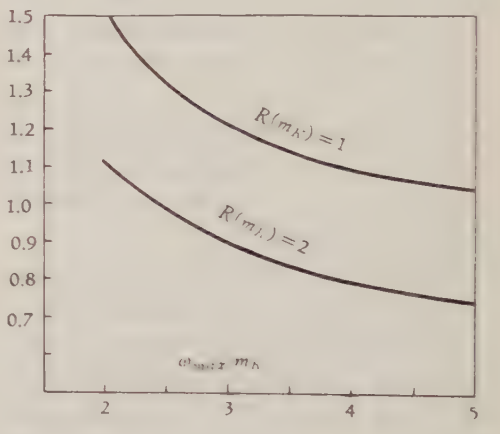

Fig. 1. Upper bound λ

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In cases of other coupling types, corresponding values of λ are not definitely positive, e. g. for the pseudoscalar type the corresponding value of λ is negative; there is no possibility for bound states to appear.
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Coupling Constant of Kaon-Baryon Interaction, II

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We have estimated in the preceding letter[1] an upper bound of the coupling constant of the kaon-baryon interactions, basing on the requirement that there exists no bound state of $K+N$. This restriction is valid but too lenient. The reason is that, if the coupling constant λ approaches to the upper bound $\hat{\lambda}$ the cross section for $K^+ + p$ scattering becomes close to $4\pi p^{-2}$ in the low energy region, which is very large compared with the experimental cross section. In the present note, for a more practical orientation about the order of magnitude of the coupling constant, we make an estimation of the cross section for the low energy $K^+ + p$ scattering and compare it with the experimental data.

The cross section is expressed as

$$\sigma(\omega_p) = 4\pi\lambda^2 \cdot \{\lambda^2 p^2 + (\omega_p + \Delta m)^2 \times (1 - \lambda \cdot A(\omega_p)^2\}^{-1} \quad (1)$$

with $\lambda = g_\Lambda{}^2 + g_\Sigma{}^2$ and

$$A(\omega_p) = \frac{\omega_p + \Delta m}{\pi} \{F_{+-}(\omega_p) + R(\omega_p) F_{-+}(\omega_p) C(\omega_p)\}. \quad (2)$$

Here we have put $\sigma_{tot} = \sigma_{el} \equiv \sigma$, since we are concerned with the low energy scattering. Eq. (1) includes all the possible virtual effects of K, $\tilde{K}$ and pions.[2] As discussed in I, $C(\omega_p)$ will presumably be small compared with other terms in eq. (2).

For given λ, a lower bound for $\sigma(\omega_p)$ can be obtained in the following way. Let $\sigma(\omega)$, obtained by dropping $C(\omega)$ and replacing $R(\omega)$ by an energy independent constant r, be denoted by $\sigma(\omega, r)$. Since $R(\omega) < 1$ and $C(\omega) > 0$, a lower bound of the cross section is given as

$$\sigma(\omega) > \sigma(\omega, 1). \quad (3)$$

If, for instance $R(\omega) \approx 2$, then

$$\sigma(\omega) \approx \sigma(\omega, 2). \quad (4)$$

$\sigma(\omega, r)$ for $r=1$ and 2, $\omega_{\max} = 2.5\, m_K$ and various values of λ are shown Figs. 1 and 2. Experiments[3] show almost constant cross section of $10 \sim 20$ mb for the kinetic energy up to about 100 Mev in the center-of-mass system. Comparing these data with Fig. 1 we see

$$\lambda < 0.7. \quad (5)$$

If $R(\omega) \approx 2$ is the case, Fig. 2 shows

$$\lambda \approx 0.6. \quad (6)$$

The result is not very sensitive to the variation of $R(\omega)$. If larger $\omega_{\max}$ is adopted, the restriction becomes more severe.

On the other hand the comparison

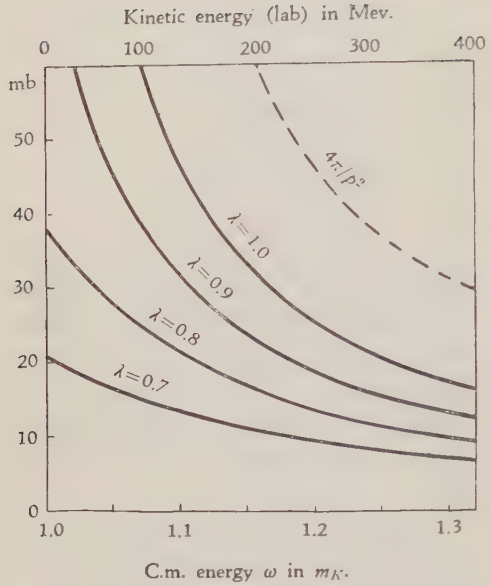

Fig. 1. $\sigma(\omega, 1)$ for $\omega_{\max}=2.5\, m_K$

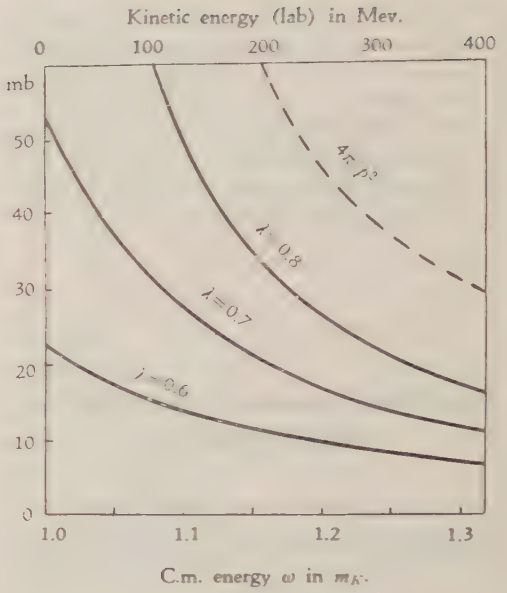

Fig. 2. $\sigma(\omega, 2)$ for $\omega_{\max}=2.5\, m_K$

between the experimental cross section for the reaction $\gamma+p\rightarrow K^{+}+\Lambda$ and the result of the lowest order perturbation calculation, based on the coupling type adopted in this note, shows $g_{\Lambda}{}^{2}=3\pm 1$[4),5)] which is obviously incompatible with eq. (5). If this large value is a reliable one, this may be taken as an evidence against the scalar type interaction. At present, however, this should not be taken too seriously since the result of the calculation is sensitive to the precise choice of anomalous magnetic moments of hyperons and moreover the very applicability of the perturbation calculation to the kaon reactions is quite doubtful.

An investigation of approximate solutions of the Low equations for the K and $\bar{K}$ scattering or capture by the nucleon has been carried out and will soon be reported.

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5) Miyachi estimated the magnitude of the coupling constants for various types of interactions applying the lowest order perturbation calculation to $K^{+}-N$ scattering and K-production by $\pi-N$ collisions. For the coupling type adopted in the present note, her estimation gives two alternatives: $0\lesssim g_{\Lambda}{}^{2}\lesssim 0.5$, $0.6\lesssim g_{\Sigma}{}^{2}\lesssim 1.8$ and $1.2\lesssim g_{\Lambda}{}^{2}\lesssim 2.3$, $0.3\lesssim g_{\Sigma}{}^{2}\lesssim 0.4$.
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A Note on the Magnetic Relaxation in a Low Field

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Manetic relaxation in a low field has been investigated by many authors[1] phenomenologically, and their descriptions can be classified into two types; (1) the so-called isotropic relaxation characterized by only one relaxation time as a result of simple modification of the usual Bloch equation, (2) the Codrington et al. relaxation[2] (hereafter abbreviated as the C. O. T. relaxation) characterized by two relaxation times T_1, T_2, longitudinal and transversal to the instantaneous magnetic field, respectively. Bloch[3] and Tomita[4] have shown that the dynamical theory can be applied to a wider range of magnetic field strength, and have obtained much more knowledge mainly in the case of (1), but they did not really attempt to understand the C. O. T. relaxation.

In these circumstances, the writer tried, along the line of Bloch's formalism,[5] to see under what conditions the C. O. T. relaxation can be specified.

In order to compare the relaxation terms of eq. (4·55) of reference 3) with the damping terms of the macroscopic C. O. T. equation[5]

$$-\frac{1}{T_1}(\boldsymbol{M}-\chi_0\boldsymbol{H}) + \frac{1}{T_3|H|^2}[\boldsymbol{H}\times(\boldsymbol{M}\times\boldsymbol{H})] \quad (1)$$

where $1/T_3 \equiv 1/T_1 - 1/T_2$, we expand the former in terms of $\sin\theta$, $\cos\theta$, where θ is an angle between the instantaneous field and z-axis in the rotating system. The results may be written for the expectation values of the spin components $\langle I_x^*\rangle$, $\langle I_y^*\rangle$ and $\langle I_z^*\rangle$

$$-(\xi'+\zeta)(\langle I_x^*\rangle - C_x^*) - \{\eta\cos^2\theta\langle I_x^*\rangle - \eta\sin\theta\cos\theta\langle I_z^*\rangle\}, \quad (2a)$$

$$-(\xi'+\zeta)\langle I_y^*\rangle - \{\eta\cos^2\theta\langle I_y^*\rangle + (\xi-\zeta)\sin^2\theta\langle I_y^*\rangle\}, \quad (2b)$$

$$-2\zeta(\langle I_z^*\rangle - C_z^*) - \{(\xi-\zeta)\sin^2\theta\langle I_z^*\rangle - (\xi-\zeta)\sin\theta\cos\theta\langle I_x^*\rangle\}, \quad (2c)$$

each referring to the rotating coordinate system by using Bloch's notation. Here ξ, ξ', η and ζ are also expressed in terms of the same Φ_μ as in (4·40) in the following way,

$$\begin{aligned}
&\xi \equiv \Phi_0(\omega)\cosh\kappa, \quad \xi' \equiv \Phi_0(\omega')\cosh\kappa', \\
&\eta \equiv \Phi_0(0) - \Phi_0(\omega')\cosh\kappa', \\
&\zeta \equiv \Phi_1(\omega+\omega')\cosh(\kappa+\kappa') + \Phi_1(\omega-\omega')\cosh(\kappa-\kappa')
\end{aligned} \quad (3)$$

respectively, where $\kappa=\hbar\omega/2kT$, $\kappa'=\hbar\omega'/2kT$, and $\omega'=\{(\omega_0-\omega)^2+\omega_1^2\}^{1/2}$.

It will readily be seen that, when $\eta\approx\xi-\zeta$ and $\xi'\approx\zeta$ are satisfied in (2), the generalized expressions (2) are reduced to the C. O. T. relaxation (1) as rewritten referring to the rotating coordinate system using the same field quantity as in (2)

$$-\frac{1}{T_1}(\boldsymbol{M}^*-\chi_0\boldsymbol{H}^*) + \frac{1}{T_3|H^*|^2}[\boldsymbol{H}^*\times(\boldsymbol{M}^*\times\boldsymbol{H}^*)], \quad (1')$$

where $\boldsymbol{H}^*$ is the magnetic field in the

rotating system. But the conditions under which $\eta \approx \tilde{\xi} - \zeta$ and $\tilde{\xi}' \approx \zeta$ are satisfied cannot be explained without our knowledge about the behaviors of Φ_μ. Such a generalized form Φ_μ as in (4·40) being of less advantage, we make use of the asymptotic behaviors of Φ_μ assuming that the correlation functions of the local field spectrum may be written in a simple exponential form. The results may be written as $\Phi(\omega_s) \approx |\Gamma|/\omega^*$ (for $\omega_s \ll \omega^*$), or $\Phi(\omega_s) \approx \omega^* |\Gamma|/\omega_s^2$ (for $\omega^* \ll \omega_s$), using a characteristic frequency ω^* of the lattice system. We can then obtain a condition under which the C. O. T. relaxation may exist;

$$\omega' (= \{(\omega_0 - \omega)^2 + \omega_1^2\}^{1/2}) \gg \omega^* \gg \omega, \quad (4)$$

from the fact that η and $\tilde{\xi} - \zeta$ do not vanish in order to be distinguished one relaxation time from the other. In this case, $\eta \approx \Phi_0(0)\{1 - (\omega^*/\omega')^2\}$, $\tilde{\xi} - \zeta \approx \Phi_0(\omega)\{1 - (\omega^*/|\omega' \pm \omega|)^2\}$, and then we have $\eta \approx \tilde{\xi} - \zeta$, as expected, because $\Phi_0(0) \approx \Phi_0(\omega)$ for $\omega^* \gg \omega$.

By comparing (2) with (1′), we obtoin

$$\Phi_0(0) - \Phi_0(\omega') \cosh \kappa' = -1/T_3 \equiv 1/T_2 - 1/T_1, \quad (5)$$

under the condition (4).

On the other hand, the coefficients $\tilde{\xi}'$ and ζ of (2) which are independent of field quantities become, in this case,

$$\tilde{\xi}' + \zeta \approx 2\zeta = 2\Phi_0(\omega') \cosh \kappa' = 1/T_1, \quad (6)$$

where use has been made of the relation $\Phi_1(\omega' \pm \omega) \simeq \Phi_1(\omega') = \Phi_0(\omega')/2$. Hence, from (5) and (6) two relaxation times T_1 and T_2 introduced phenomenologically by Codrington et al. are given by

$$1/T_1 = 2\Phi_0(\omega') \cosh \kappa',$$

$$1/T_2 = \Phi_0(0) + \Phi_0(\omega') \cosh \kappa'. \quad (7)$$

Finally, the writer want to emphasize that the condition (4) is difficult to attain experimentally, because it means an extreme off-resonance or strong saturation for the field quantities ω, ω_1, and ω_0. However, a possible way is to keep a sample in such a way that ω^* may satisfy the condition (4) under a given external magnetic field.

In the usual experimental condition ($\omega^* \gg \omega'$, ω), it is found that $1/T_1 \approx 1/T_2 \approx 2\Phi_0(0) \cosh \kappa'$ because of the relations $\Phi_0(\omega) \approx \Phi_0(\omega') \approx 2\Phi_1(\omega + \omega') \approx \Phi_0(0)$. This is the previously mentioned case of isotropic relaxation, and should be observed as the main feature of the magnetic relaxation in a low field.

1) For instance, M. A. Garstens and J. I. Kaplan, Phys. Rev. **99** (1955), 459.
2) Codrington, Olds, and Torrey, Phys. Rev. **95** (1954), 607.
3) F. Bloch, Phys. Rev. **105** (1957), 1206.
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A Remark on the Anomalous Magnetic Moment in the Static Model

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June 23, 1958

In the static model of pion-nucleon theory, the interaction Hamiltonian H_{NM} is given by

$$H_{NM}=\frac{f_0}{\mu}\int\rho(r)\,\sigma\nabla\phi_\alpha\cdot\tau_\alpha\,d\boldsymbol{r}. \qquad (1)$$

When an electromagnetic field is involved, an additional Hamiltonian H_{NMR} is required by gauge invariance.

$$H_{NMR}=-e\frac{f_0}{\mu}\int\rho(r)\,\sigma\boldsymbol{A}\,(\tau_1\phi_2-\tau_2\phi_1)\,d\boldsymbol{r}. \qquad (2)$$

This interaction is usually neglected for the calculation of the anomalous magnetic moment since the relativistic perturbation theory shows that the contribution from the corresponding interaction is negligible.

However, if the source function $\rho(r)$ is extended, this Hamiltonian, which seems to give only a small contribution to the magnetic moment anomaly, gives rise to an anomalous magnetic moment

$$\begin{aligned}M_{i\,t}=&-1/(4\sqrt{6\pi R})\\&\times ef_0/\mu\sum\frac{k}{\sqrt{2\omega_k}}\frac{dv(k)}{dk}\\&\times[\sigma_+\tau_+(a_{1,1}+b^*_{1,-1})\\&+\sigma_-\tau_-(b_{1,-1}+a^*_{1,1})\\&-\sigma_+\tau_-(b_{1,1}+a^*_{1,-1})\\&-\sigma_-\tau_+(a_{1,-1}+b^*_{1,1})]\end{aligned} \qquad (3)$$

where $v(k)$ is a cutoff function. This contributes only to the vector part of the magnetic moment, so we need to calculate the expectation value of this operator only for proton. Making use of the technique developed by Miyazawa,[1] we get the following result (in nuclear magneton).

$$\begin{aligned}\langle|M_{\rm int}|\rangle=&\frac{f^2}{4\pi}\cdot\frac{8}{3\pi}\frac{M}{\mu^2}\\&\times\int\frac{k^3}{\omega_k^2}v(k)\frac{dv(k)}{dk}dk\\&+\frac{2}{27\pi^3}M\iint\frac{k^3\sigma_{33}(l)\,v(k)}{\omega_k\omega_l(\omega_k+\omega_l)[v(l)]^2}\\&\times\frac{dv(k)}{dk}dk\,dl\\=&-\frac{f^2}{4\pi}\cdot\frac{4}{3\pi}\frac{M}{\mu^2}\\&\times\int\left[\frac{3k^2}{\omega_k^2}-\frac{2k^4}{\omega_k^4}\right]v(k)^2dk\\&-\frac{1}{27\pi^3}M\iint\left[\frac{3k^2}{\omega_k\omega_l(\omega_k+\omega_l)}\right.\\&\left.-\frac{(2\omega_k+\omega_l)k^4}{\omega_k^3\omega_l(\omega_k+\omega_l)^2}\right]\\&\times\sigma_{33}(l)\frac{v^2(k)}{v^2(l)}dk\,dl\end{aligned} \qquad (4)$$

The last integral is obtained by a partial integration assuming that the cutoff function vanishes at high momenta. We also assume the contribution from the $(3\cdot3)$-state to be dominant. Numerical values are as follows.

$$\langle|M_{\rm int}|\rangle=-1.12-0.15=-1.27 \quad (K=5\mu)$$

$$\langle|M_{\rm int}|\rangle=-1.34-0.20=-1.54 \quad (K=6\mu).$$

The first numbers in these equations represent the first terms in (4) and the second numbers are the remaining terms. Adding these to Miyazawa's results, we obtain

$$\mu_v=1.54-0.18-1.27=0.09 \quad (K=5\mu)$$

$$\mu_v=2.05-0.18-1.54=0.33 \quad (K=6\mu).$$

Compared with the experimental value, $\mu_v=1.85$, this result is extremely small. It has been believed that the vector part of the magnetic moment anomaly belongs

to a low energy phenomenon to which the static model has been successfully applied. If so, this result calculated by the static model seems very difficult to explain ; there seems to exist nothing to compensate this large discrepancy. It may be noticed, however, that the static model yields a fairly good result as was shown by Miyazawa, if we neglect M_{int}. The difficult situation we have encountered may be avoided, in our opinion, in the following way.

In the static model, the definition of charge and current densities becomes essentially ambiguous. Indeed, in such a theory it is not possible to fulfill the continuity equation everywhere inside the source. It is fulfilled only outside the source, and inside the source it can be fulfilled only in the form averaged over a volume V in which the source is included, i. e.[2]

$$\int_v \left[\frac{\partial S_0}{\partial t} + \operatorname{div} \boldsymbol{S} \right] dv = 0.$$

Expressions for S_0 and $\boldsymbol{S}$ which fulfill this weaker condition and gauge invariance does not seem to be unique. An additional appropriate current may contribute to the magnetic moment anomaly which cancels the contribution from M_{int}, although we have not succeeded in finding them.

On the other hand, if we take the standpoint that the static model must be regarded as a low energy limit of the relativistic theory, we may have to neglect M_{int} in our case.

Finally, we remark on the anomalous magnetic moment of the nucleus.[4] In a nucleus the propagator is not $\delta(x-y)$ but $\delta(x-y)-\rho(x-y)$, according to the Pauli principle, so the interaction becomes effectively an extended one, and we cannot neglect M_{int}. However, we should not use the same $\rho(r)$ in H_{NM} and H_{NMR} since their origins are not the same.

The authors would like to thank Dr. Miyazawa for suggesting this problem.

After the completion of this work, we happened to know that R. H. Capps and W. G. Holladay had treated a problem which is essentially the same with ours.[3] But their conclusion is different from ours ; and it seems that their treatment is inconsistent when seen from the standpoint of the static extended source pion theory, since the probability P_1^c becomes very large if we use $f^2/4\pi \sim 0.08$ and $K=5\mu \sim 6\mu$ though they assume P_1^c to be 10%. If we put P_1^c to be about 60%$\sim$100% in their paper, we find $\langle M_z \rangle$ to be about $\mp 0.78 \sim \mp 1.3$, which is consistent with our result.

1) H. Miyazawa, Phys. Rev. **101** (1956), 1564.
2) W. Pauli and S. M. Dancoff, Phys. Rev. **62** (1942), 85.
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